当 $n \geq N$ 时 $|a_{n+2}-a_n| = C \in Z_+$. C 为常数

1° 当 ... 时 $|a_{n+2}+1| = 1$ $\therefore a_{n+2} \in \{0, -2\}$

2° 对 $n \geq N+2$ 则 $a_n, a_{n+1}, a_{n+2} \in \{0, -2\}$.

不满足①.

12. $f(1)=f(2)=1$. $f(3)=2$.

(1) 若 $a_2=2$. 设 $b_i = a_{i+1}-1$ $(i=1,2\cdots n-1)$ 则 $b_1, b_2 \cdots b_{n-1}$ 是 $1, 2 \cdots n-1$ 之一个排列. $b_1=1$. 且 $|b_i - b_{i-1}| = |a_{i+1}-a_i| \leq 2$ $\therefore$ 满足条件的排列有 $f(n-1)$ 个

(2) 若 $a_2=3$ (i) 当 $a_3=2$ 时 $a_4=4$. 设 $c_i = a_{i+3}-3$ $(i=1,2\cdots n-3)$ 则 $c_1, c_2, \cdots c_{n-3}$ 是 $1, 2 \cdots n-3$ 之一个排列 且 $|c_i - c_{i-1}| = |a_{i+3}-a_{i+2}| \leq 2$ $\therefore$ 满足条件的排列有 $f(n-3)$ 个

(ii) $a_3=5,\cdots,a_k=2k-1$，a_{k+1} 为偶数

若 $a_{k+1}=2k$ $(k\geq 2)$，若 $a_{k+2}=2k+1$ 或 $2k+2$，则在排列中不出现 2，矛盾。∴ $a_{k+2}=2k-2$，……，$a_{2k}=2$，此时 $n=2k$，这样的排列有1个

若 $a_{k+1}=2k-2$ $(k\geq 3)$，若 $a_{k+2}=2k$，则在排列中不出现 2，矛盾，$a_{k+2}=2k-4$，……，$a_{2k-1}=2$，此时 $n=2k-1$，这样的排列有1个

∴ $f(n)=f(n-1)+f(n-3)+1$ $(n\geq 4)$

∴ $f(n)\equiv 1,1,2,0,2,1,2,1,3,2,0,0,3,0,1,1,2 \pmod 4$

$f(n)$ 模4的周期为14，∵ $2024\equiv 8 \pmod{14}$，

∴ $f(2024)\equiv f(8)\equiv 1 \pmod 4$

13 ∵ $a_n(b_n+b_{n+2}) = b_{n+1}$

∴ $a_n(b_{n+2}^2 - b_n^2) = a_n(b_n+b_{n+2})(b_{n+2}-b_n) = b_{n+1}(b_{n+2}-b_n)$

$= -\frac{1}{2}\left((b_{n+1}-b_{n+2})^2 - (\cancel{b_{n+1}^2}+b_{n+2}^2)\right) + \frac{1}{2}\left((b_n-b_{n+1})^2-(b_n^2+\cancel{b_{n+1}^2})\right)$

$(2a_n-1)(b_{n+2}^2-b_n^2) = (b_n-b_{n+1})^2-(b_{n+1}-b_{n+2})^2$ ①

∵ $2a_n-1 \geq 2a_0-1 > 2\times\frac{1}{2} - 1 = 0$

∴ 由①可得 $b_{n+2}^2-b_n^2 = \frac{(b_n-b_{n+1})^2-(b_{n+1}-b_{n+2})^2}{2a_n-1}$ ② 16

在②中用0, 1, ⋯ n-2 代替n, 再求和

$b_n^2+b_{n-1}^2-b_0^2-b_1^2 = \frac{(b_0-b_1)^2}{2a_0-1} - \frac{(b_1-b_2)^2}{2a_0-1} + \frac{(b_1-b_2)^2}{2a_1-1} - \frac{(b_2-b_3)^2}{2a_1-1} + \cdots + \frac{(b_{n-2}-b_{n-1})^2}{2a_{n-2}-1} - \frac{(b_{n-1}-b_n)^2}{2a_{n-2}-1}$

$= \frac{(b_0-b_1)^2}{2a_0-1} - (b_1-b_2)^2\left(\frac{1}{2a_0-1}-\frac{1}{2a_1-1}\right) - \cdots - (b_{n-2}-b_{n-1})^2\left(\frac{1}{2a_{n-3}-1}-\frac{1}{2a_{n-2}-1}\right) - \frac{(b_{n-1}-b_n)^2}{2a_{n-2}-1}$

$$\le \frac{(b_0-b_1)^2}{2a_0-1}$$

$\therefore b_n^2 \le b_n^2+b_{n-1}^2 \le b_0^2+b_1^2+\frac{(b_0-b_1)^2}{2a_0-1} \triangleq |M|^2$

$\therefore |b_n| \le |M|$

14. 分析：假设 $\{x_n\}$ 是常数列，设 $x_n=a>0$ 且满足条件

则 $a=\sqrt{\alpha a-a}$ $\therefore a^2=a(\alpha-1)$ $\therefore a=\alpha-1>0$ $\therefore \alpha>1$.

若 $\alpha>1$，则取 $x_n=\alpha-1>0$ 且满足 $\sqrt{\alpha(\alpha-1)-(\alpha-1)}=\alpha-1$，满足条件

若 $\alpha\le 1$ 且存在满足条件的数列.

$\therefore \alpha x_{n+1}-x_n>0$ $\therefore x_{n+1}\ge \alpha x_{n+1}>x_n$ $\therefore \{x_n\}$ 严格递增

$\therefore x_{n+2}^2=\alpha x_{n+1}-x_n \overset{①}{\le} x_{n+1}-x_n<x_{n+1}$

$\therefore x_{n+1}^2\le x_{n+2}^2<x_{n+1}$ $\therefore x_{n+1}<1$

在①中用 $1, 2, \cdots, n$ 代替 n，再求和可得

$nx_3^2 < x_3^2 + x_4^2 + \cdots + x_{n+2}^2 \leq x_{n+1} - x_1 < x_{n+1}$ ②

当 $n \to +\infty$ ③ 与 $x_{n+1} < 1$ 矛盾

15 $x_{n+2} - (k+1)x_{n+1} + x_n = 0$ 特征方程为 $\lambda^2 - (k+1)\lambda + 1 = 0$ ①

若 $\lambda = 1$ 则 $1 - k - 1 + 1 = 0$ $k = 1$ 矛盾 $\therefore \lambda \neq 1$

设①两根为 α, $\frac{1}{\alpha}$ 且 α 的对称性设 $\alpha > 1$.

由韦达定理可得 $\alpha + \frac{1}{\alpha} = k+1$ $\qquad \alpha = \frac{k+1+\sqrt{(k+1)^2-4}}{2}$

设 $x_n = C_1\alpha^n + C_2\alpha^{-n}$

$\because x_0 = 1 = C_1 + C_2$ $\qquad \therefore C_1 = \frac{\alpha}{\alpha-1}$ $\quad C_2 = -\frac{1}{\alpha-1}$

$\alpha + \frac{1}{\alpha} + 1 = k+2 = x_1 = C_1\alpha + C_2 \cdot \frac{1}{\alpha}$

$\therefore x_n = \frac{\alpha^{n+1}}{\alpha-1} - \frac{\alpha^{-n}}{\alpha-1}$ ②

夸克扫描王
极速扫描，就是高效

$$\therefore \alpha = \frac{2k+2+2\sqrt{(k-1)(k+3)}}{4} = \left(\frac{\sqrt{k-1}+\sqrt{k+3}}{2}\right)^2 \triangleq \beta^2.$$

代入② 可得 $x_n = \dfrac{\beta^{2n+2}-\beta^{-2n}}{\beta^2-1} = \dfrac{\beta^{2n+1}-\beta^{-(2n+1)}}{\beta-\beta^{-1}}$ ③

设 $k=m^2-3$ $(m\in \mathbb{Z}_+,\ m\geq 3)$ 则

$$\beta = \frac{\sqrt{k-1}+\sqrt{k+3}}{2} = \frac{\sqrt{m^2-4}+m}{2}$$

$$= \frac{2\sqrt{(m-2)(m+2)}+2m}{4} = \left(\frac{\sqrt{m-2}+\sqrt{m+2}}{2}\right)^2 \triangleq \gamma^2$$

代入③ 可得 $x_n = \dfrac{\gamma^{2(2n+1)}-\gamma^{-2(2n+1)}}{\gamma^2-\gamma^{-2}} = \dfrac{\gamma^{2n+1}-\gamma^{-(2n+1)}}{\gamma-\gamma^{-1}} \cdot \dfrac{\gamma^{2n+1}+\gamma^{-(2n+1)}}{\gamma+\gamma^{-1}}$

$\triangleq y_n \cdot z_n$ $\quad \therefore y_n = \dfrac{\gamma}{\gamma-\gamma^{-1}} \cdot \beta^n - \dfrac{\gamma^{-1}}{\gamma-\gamma^{-1}} \cdot \beta^{-n}$.

$y_{n+2} - m\,y_{n+1} + y_n = 0$ ④ $\quad y_0 = 1.\quad y_1 = m+1.$

同理 $z_{n+2} - m\,z_{n+1} + z_n = 0$ ⑤ $\quad z_0 = 1 \quad z_1 = m-1 \geq 2$

由④⑤可知 $y_n, z_n \in \mathbb{Z}_+$ 且当 $n \in \mathbb{Z}_+$ 时 $z_n \geq 2$

∴ x_n 是分数　$x_0 = 1$ 不是分数

16. $x_{n+1} = \dfrac{x_n^2}{x_n^2 - x_n + 1} = 1 + \dfrac{x_n - 1}{x_n^2 - x_n + 1}$　∴ $x_{n+1} - 1 = \dfrac{x_n - 1}{x_n^2 - x_n + 1}$ ①

∵ $x_n^2 - x_n + 1 > 0$ 由① $x_n - 1$ 与 $x_{n+1} - 1$ 同号　∵ $x_1 = \frac{4}{3} > 1$

∴ 对于 $n \in \mathbb{Z}_+$ 都有 $x_n > 1$

由①可得 $\dfrac{1}{x_{n+1} - 1} = \dfrac{x_n(x_n - 1) + 1}{x_n - 1} = x_n + \dfrac{1}{x_n - 1}$

∴ $x_n = \dfrac{1}{x_{n+1} - 1} - \dfrac{1}{x_n - 1}$ ②　在②中用 $1, 2 \cdots n$ 代替 n

并求和可得 $\sum_{k=1}^{n} x_k = \dfrac{1}{x_{n+1} - 1} - \dfrac{1}{x_1 - 1} = \dfrac{1}{x_{n+1} - 1} - \dfrac{1}{\frac{4}{3} - 1} = \dfrac{1}{x_{n+1} - 1} - 3$

$= \dfrac{x_n(x_n - 1) + 1}{x_n - 1} - 3 = \dfrac{x_n^2 - 4x_n + 4}{x_n - 1} = \dfrac{(x_n - 2)^2}{x_n - 1}$ ③

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$\because x_n \in \mathbb{Q}_+$ $\therefore x_n - 1 \in \mathbb{Q}_+$ 设 $x_n - 1 = \frac{a}{b}$ $a, b \in \mathbb{Z}_+$, $(a,b)=1$

代入③ 可得 $\sum_{k=1}^{n} x_k = \frac{\left(\frac{a}{b}-1\right)^2}{\frac{a}{b}} = \frac{(a-b)^2}{ab}$ ④

$\because (a-b, a) = (a-b, b) = 1$ $\therefore$ ④应为最简分数，分子 $(a-b)^2$ ✓

17.(1) 设 $S = \min\{a_{n+1} - a_n \mid n \in \mathbb{Z}_+\} \in \mathbb{Z}_+$ 存在 $k \in \mathbb{Z}_+$ 使得 $2^k > p$

设 $a_{m+1} - a_m = S$ 其中 $m \in \mathbb{Z}_+$ 则

$$2(a_{2^{k-1}(m+1)} - a_{2^k m}) = a_{2^k(m+1)} - a_{2^k m} = \underline{a_{2^k(m+1)} - a_{2^k(m+1)-1}} + \cdots + \underline{a_{2^k m+1} - a_{2^k m}}$$

$$\geq S + \cdots + S = 2^k \cdot S$$

$$= \cdots = 2^k(a_{m+1} - a_m) = 2^k \cdot S$$

等号成立

即 $a_{2^k(m+1)} - a_{2^k(m+1)-1} = \cdots = a_{2^k m+1} - a_{2^k m} = S$

$T \triangleq \{a_{2^k m+i} \mid i=0,1,\cdots,2^k\}$ 为等差数列，公差为 S

设 $T'=\{a_{2^k m+i} \mid i=0,1,\cdots,p-1\}$ 为等差数列，公差为 S

假设存在 $0\le i<j\le p-1$ 使得 $a_{2^k m+j}\equiv a_{2^k m+i} \pmod p$

则 $\underline{(j-i)}S\equiv 0 \pmod p$ ∵ $0<j-i<p$ ∴ $S\equiv 0 \pmod p$

∴ $0<S\le a_2-a_1=2a_1-a_1=a_1<p$ 矛盾

∴ T' 是模 p 的完全剩余系 ∴ T' 中有一项是 p 的倍数

(2) 对于给定 $n\in\mathbb{Z}_+$ 存在唯一的非负整数 k_n，使得 $2^{k_n}\le n<2^{k_n+1}$

∵ $2^{k_{2n}}\le \underline{2n}<2^{k_{2n}+1}$ 且 $2^{k_n+1}\le \underline{2n}<2^{k_n+2}$ ∴ $k_{2n}=k_n+1$

且 $k_n\le k_{n+1}$ 设 $a_n=2^{k_n}+np$ ∴ 数列 $\{a_n\}$ 严格递增

∵ $a_{2n}=2^{k_{2n}}+2np=2^{k_n+1}+2np=2(2^{k_n}+np)=2a_n$

∵ $\{a_n\}$ 是严格增数列 ∴ $P \nmid a_n$ $(n\in\mathbb{Z}_+)$

18. 设 $r_i=\sqrt{\sigma(i)+\sqrt{\cdots+\sqrt{\sigma(n)}}}>0$ $(i=1,2\cdots n)$ ∵ $r_1\in\mathbb{Q}_+$

∴ $r_2=r_1^2-\sigma(1)\in\mathbb{Q}_+$, …… $r_n=r_{n-1}^2-\sigma(n-1)\in\mathbb{Q}_+$

∴ $r_n=\sqrt{\sigma(n)}\in\mathbb{Z}_+$ $r_{n-1}=\sqrt{\sigma(n-1)+r_n}\in\mathbb{Z}_+$ … $r_1=\sqrt{\sigma(1)+r_2}\in\mathbb{Z}_+$

下证对每个 $i=1,2\cdots n$ 都有 $r_i<\sqrt{n}+1$ ①

$i=n$ 时 $r_n=\sqrt{\sigma(n)}\le\sqrt{n}<\sqrt{n}+1$

假设 $r_{i+1}<\sqrt{n}+1$ 则 $r_i=\sqrt{\sigma(i)+r_{i+1}}<\sqrt{n+\sqrt{n}+1}<\sqrt{n}+1$

∴ ①成立

对于正整数 n 存在唯一的正整数 l 使得 $l^2\le n<(l+1)^2$

设 $\sigma(i)=l^2$ $i\in\{1,2\cdots n\}$

若 $i<n$，则 $l<r_i=\sqrt{\sigma(i)+r_{i+1}}=\sqrt{l^2+r_{i+1}}<\sqrt{n}+1<l+2$

$\therefore r_i = l+1 \quad \therefore l^2 + r_{i+1} = l^2 + 2l + 1$

$\therefore 2l+1 = r_{i+1} < \sqrt{n}+1 < l+2 \Rightarrow l<1$ 矛盾 $\therefore i=n$

即 $\sigma(n) = l^2$

若 $l \geq 2$，则存在 $j \in \{1,2,\cdots,n-1\}$ 使得 $\sigma(j) = l^2-1$

$\therefore l < r_j = \sqrt{\sigma(j) + r_{j+1}} = \sqrt{l^2 - 1 + r_{j+1}} < \sqrt{n}+1 < l+2$

$r_{j+1} > 1$

$\therefore r_j = l+1 \quad \therefore l^2 - 1 + r_{j+1} = l^2 + 2l + 1$

$\therefore 2l+2 = r_{j+1} < \sqrt{n}+1 < l+2 \Rightarrow l<0$ 矛盾

$\therefore l = 1 \quad 1 \leq n < 4 \quad n=1,\ \sqrt{1}$ ✓

$n=2,\ \sqrt{1+\sqrt{2}},\ \sqrt{2+\sqrt{1}}$ ✗

$n=3\ \sqrt{2+\sqrt{3+\sqrt{1}}}$ ✓

19. 不妨假设 $x \geq y \geq z$ 则 $x^2-2(y+z)x+(y-z)^2=0$

$$x=\frac{2(y+z)\pm\sqrt{[2(y+z)]^2-4(y-z)^2}}{2}=y+z\pm2\sqrt{yz}=(\sqrt{y}\pm\sqrt{z})^2$$

$\therefore \sqrt{x}=\sqrt{y}\pm\sqrt{z}$ $\therefore \sqrt{x}=\sqrt{y}+\sqrt{z}$

$\therefore$ 原不等式 $\Leftrightarrow \dfrac{(\sqrt{y}+\sqrt{z})^2+y+z}{3}\geq\sqrt[3]{2(\sqrt{y}+\sqrt{z})^2\cdot yz}$ ①

若 $y=0$ 则 ① 成立.

若 $y>0$ 设 $t=\frac{z}{y}$ 则 ① 化为 $\dfrac{2(t+1)+2\sqrt{t}}{3}\geq\sqrt[3]{2t(\sqrt{t}+1)^2}$ ② 而

② 左 $=\dfrac{2t+\sqrt{t}+1+\sqrt{t}+1}{3}\geq\sqrt[3]{2t\cdot(\sqrt{t}+1)(\sqrt{t}+1)}=$ ② 右

均值不等式

20 (1) 若 m 为偶数 设 $m=2k$ $(k\in\mathbb{Z}^+)$ 不妨设 $a_1\leq a_2\leq\cdots\leq a_{2k}$

设 $A_1=\dfrac{\sum_{i=1}^{k}a_i}{k}$ $A_2=\dfrac{\sum_{i=k+1}^{2k}a_i}{k}$ 则 $A_1\leq10\leq A_2$

若 $A_1=10$ 则 $A_2=10$ $\therefore a_i=10$ $(i=1,2\cdots 2k)$ 矛盾

$\therefore A_1<10$ 同理 $A_2>10$

设 $A_1=10-x$ $x>0$ 则 $A_2=10+x$.

若 $x<1$ 则 $A_1>9$ $A_2<11$

$\therefore a_k\geq 11$ $\therefore a_{2k}\geq\cdots\geq a_{k+1}\geq a_k\geq 11$ $\therefore A_2\geq 11$ 矛盾

$\therefore x\geq 1$ $\therefore \left(\prod_{i=1}^{2k}a_i\right)^{\frac{1}{2k}}=\sqrt{\left(\prod_{i=1}^{k}a_i\right)^{\frac{1}{k}}\left(\prod_{i=k+1}^{2k}a_i\right)^{\frac{1}{k}}}$

$\leq\sqrt{A_1A_2}=\sqrt{(10-x)(10+x)}=\sqrt{100-x^2}\leq\sqrt{100-1}=3\sqrt{11}$.

均值不等式

(2) 若 m 为奇数 $\sum_{i=1}^{m}a_i+\sum_{i=1}^{m}a_i=10\cdot 2m$. $\therefore$ 对 $2m$. 由 (1)

可得 $\left(\prod_{i=1}^{m}a_i\right)^{\frac{1}{m}}=\left(\prod_{i=1}^{m}a_i\prod_{i=1}^{m}\right)^{\frac{1}{2m}}\leq 3\sqrt{11}$

21、 $$\frac{a+\sqrt{ab}+\sqrt[3]{abc}+\sqrt[4]{abcd}}{4} \le \sqrt[4]{a\cdot\frac{a+b}{2}\cdot\frac{a+b+c}{3}\cdot\frac{a+b+c+d}{4}} \quad ①$$

$$① \Leftrightarrow \frac{1}{4}\left(\sqrt[4]{\frac{a}{a}\cdot\frac{2a}{a+b}\cdot\frac{3a}{a+b+c}\cdot\frac{4a}{a+b+c+d}}+\sqrt[4]{\frac{a}{a}\cdot\frac{2a}{a+b}\cdot\frac{3b}{a+b+c}\cdot\frac{4b}{a+b+c+d}}\right.$$
$$\left.+\sqrt[4]{\frac{a}{a}\cdot\frac{2b}{a+b}\cdot\frac{\sqrt[3]{abc}}{\frac{a+b+c}{3}}\cdot\frac{4c}{a+b+c+d}}+\sqrt[4]{\frac{a}{a}\cdot\frac{2b}{a+b}\cdot\frac{3c}{a+b+c}\cdot\frac{4d}{a+b+c+d}}\right)\le 1. \quad ②$$

由均值不等式可得

$$② 左 \le \frac{1}{4}\cdot\frac{1}{4}\left(\frac{a}{a}+\frac{2a}{a+b}+\frac{3a}{a+b+c}+\frac{4a}{a+b+c+d}+\frac{a}{a}+\frac{2a}{a+b}+\frac{3b}{a+b+c}+\frac{4b}{a+b+c+d}+\frac{a}{a}+\frac{2b}{a+b}+\frac{3c}{a+b+c}+\frac{4d}{a+b+c+d}\right.$$
$$\left.+\frac{a}{a}+\frac{2b}{a+b}+\frac{\sqrt[3]{abc}}{\frac{a+b+c}{3}}+\frac{4c}{a+b+c+d}\right)$$

$$=\frac{1}{16}\left(4+4+3+4+\frac{\sqrt[3]{abc}}{\frac{a+b+c}{3}}\right)\le\frac{1}{16}(15+1)=1.$$

均值不等式