21 $$\frac{a+\sqrt{ab}+\sqrt[3]{abc}+\sqrt[4]{abcd}}{4} \le \sqrt[4]{a\cdot\frac{a+b}{2}\cdot\frac{a+b+c}{3}\cdot\frac{a+b+c+d}{4}} \quad ①$$

$$① \Leftrightarrow \frac{1}{4}\left(\sqrt[4]{\frac{a}{a}\cdot\frac{2a}{a+b}\cdot\frac{3a}{a+b+c}\cdot\frac{4a}{a+b+c+d}}+\sqrt[4]{\frac{a}{a}\cdot\frac{2a}{a+b}\cdot\frac{3b}{a+b+c}\cdot\frac{4b}{a+b+c+d}}\right.$$
$$\left.+\sqrt[4]{\frac{a}{a}\cdot\frac{2b}{a+b}\cdot\frac{\sqrt[3]{abc}}{\frac{a+b+c}{3}}\cdot\frac{4c}{a+b+c+d}}+\sqrt[4]{\frac{a}{a}\cdot\frac{2b}{a+b}\cdot\frac{3c}{a+b+c}\cdot\frac{4d}{a+b+c+d}}\right)\le 1. \quad ②$$

由均值不等式得

$$② 左 \le \frac{1}{4}\cdot\frac{1}{4}\left(\frac{a}{a}+\frac{2a}{a+b}+\frac{3a}{a+b+c}+\frac{4a}{a+b+c+d}+\frac{a}{a}+\frac{2a}{a+b}+\frac{3b}{a+b+c}+\frac{4b}{a+b+c+d}+\frac{a}{a}+\frac{2b}{a+b}+\frac{3c}{a+b+c}+\frac{4d}{a+b+c+d}\right.$$
$$\left.+\frac{a}{a}+\frac{2b}{a+b}+\frac{\sqrt[3]{abc}}{\frac{a+b+c}{3}}+\frac{4c}{a+b+c+d}\right)$$
$$=\frac{1}{16}\left(4+4+3+4+\frac{\sqrt[3]{abc}}{\frac{a+b+c}{3}}\right)\underset{均值不等式}{\le}\frac{1}{16}(15+1)=1.$$

22 设 $a_i^2=\tan x_i$ $(i=1,2\cdots n)$ $x_i\in\left(0,\frac{\pi}{2}\right)$

则 $\sum_{i=1}^{n}\frac{1}{1+a_i^4}=1$ 化为 $\sum_{i=1}^{n}\frac{1}{1+\tan^2 x_i}=1$ 即 $\sum_{i=1}^{n}\cos^2 x_i=1$ ①

$a_1a_2\cdots a_n \geq (n-1)^{\frac{n}{2}}$ 化为 $\tan x_1 \cdots \tan x_n \geq (n-1)^{\frac{n}{2}}$ ②

对每个 $j=1,2\cdots n$

$$\sin^2 x_j = 1-\cos^2 x_j \overset{①}{=} \sum_{\substack{i=1\\i\neq j}}^{n} \cos^2 x_i \geq (n-1)\sqrt[n-1]{\prod_{\substack{i=1\\i\neq j}}^{n} \cos^2 x_i}.$$

由均值不等式

对所有 $j=1,2\cdots n$，可得

$$\prod_{j=1}^{n} \sin^2 x_j \geq (n-1)^n \prod_{j=1}^{n} \cos^2 x_j \Rightarrow \prod_{j=1}^{n} \tan^2 x_j \geq (n-1)^n$$

∴ ② 得证

23. 设 $a=\sum_{j=1}^{n}\frac{x_j}{n}$　$b=\sum_{j=1}^{n}\frac{y_j}{n}$　$a_i=x_i-a$. $b_i=y_i-b$. $(i=1,2\cdots n)$

则原不等式化为 $\left|\sum_{i=1}^{n}(a_i+a)(b_{i+1}+b)-(a_{i+1}+a)(b_i+b)\right| \leq \sum_{i=1}^{n}(a_i^2+b_i^2)$ ①

①左 $=\left|\sum_{i=1}^{n}[a_ib_{i+1}+a_ib+ab_{i+1}-(a_{i+1}b_i+a_{i+1}b+ab_i]\right|$

$$\le \left| \underbrace{\sum_{i=1}^{n}(a_i b_{i+1} - a_{i+1} b_i)} + b\Big(\underbrace{\sum_{i=1}^{n} a_i - \sum_{i=1}^{n} a_{i+1}}_{0}\Big) + a\Big(\underbrace{\sum_{i=1}^{n} b_{i+1} - \sum_{i=1}^{n} b_i}_{0}\Big) \right|$$

$$= \left| \sum_{i=1}^{n}(a_i b_{i+1} - a_{i+1} b_i) \right|$$

$$\le \sum_{i=1}^{n} |a_i b_{i+1}| + \sum_{i=1}^{n} |a_{i+1} b_i| \le \frac{1}{2}\sum_{i=1}^{n}(a_i^2 + b_{i+1}^2) + \frac{1}{2}\sum_{i=1}^{n}(a_{i+1}^2 + b_i^2)$$

$=$ ① 不2

24. 设 $x_i = \sqrt{\frac{a_{i+1}^2+1}{a_i^2+1}}$ $(i=1,2,\dots,n)$ 则 $x_i \ge 1 \Leftrightarrow a_{i+1} \ge a_i$

设 $y_i = \frac{\frac{a_{i+1}}{a_i}}{x_i}$ $(i=1,2,\dots,n)$ 则 $\prod_{i=1}^{n} y_i = 1$ ~~且~~ $y_i \ge 1$

$\Leftrightarrow \frac{a_{i+1}}{a_i} \ge \sqrt{\frac{a_{i+1}^2+1}{a_i^2+1}} \Leftrightarrow a_{i+1}^2(a_i^2+1) \ge a_i^2(a_{i+1}^2+1)$

$\Leftrightarrow a_{i+1} \ge a_i$ ∴ $x_i \ge 1 \Leftrightarrow y_i \ge 1$

∴ $x_i - 1$ 与 $y_i - 1$ 同号 因此 $(x_i-1)(y_i-1) \ge 0$

$\Rightarrow x_i(y_i-1) \geq y_i - 1$

$\therefore$ 左 - 右 $= \sum_{i=1}^{n} x_i y_i - \sum_{i=1}^{n} x_i = \sum_{i=1}^{n} x_i(y_i-1) \geq \sum_{i=1}^{n}(y_i-1)$

$= \sum_{i=1}^{n} y_i - n \underset{\text{均值不等式}}{\geq} n\sqrt[n]{\prod_{i=1}^{n} y_i} - n = n - n = 0.$

25 $\underline{x^2+y^2+z^2 \geq xy+yz+zx}$ 　由柯西不等式可得

排序不等式

$\sum \frac{x^2}{xy+z} \cdot \sum x(xy+z) \geq \underline{\left(\sum x^{\frac{3}{2}}\right)^2} \geq \frac{(\sum x)^3}{3}$ 　　30-

由幂平均不等式可得 $\left(\frac{\sum x^{\frac{3}{2}}}{3}\right)^{\frac{2}{3}} \geq \frac{\sum x}{3}$, $\therefore \left(\sum x^{\frac{3}{2}}\right)^2 \geq \frac{(\sum x)^3}{3}$,

$\therefore \sum \frac{x^2}{xy+z} \geq \frac{(\sum x)^3}{3\sum x(xy+z)} \geq \frac{(\sum x)^3}{3\left(\sum x^2 y + \sum x^2\right)} = \frac{(\sum x)^3}{3\sum x^2(y+1)}$

∵ $a_ia_j \le \frac{a_i^2+a_j^2}{2} \le \frac{n}{2}$ ∴ $n-a_ia_j \ge n-\frac{n}{2} > 0$

$(1\le i<j\le n)$ ∴ $\frac{1}{n-a_ia_j} \le \frac{1}{n-|a_i||a_j|}$.

∴ $\sum_{1\le i<j\le n}\frac{1}{n-a_ia_j} \le \sum_{1\le i<j\le n}\frac{1}{n-|a_i||a_j|} \le \frac{n}{2}$.

不妨假设 $a_i \ge 0$ $(i=1,2\dots n)$

只需证 $\sum_{1\le i<j\le n}\frac{1}{n-a_ia_j} \le \frac{n}{2} \Leftrightarrow \sum_{1\le i<j\le n}\frac{n}{n-a_ia_j} \le \frac{n^2}{2}$

$\Leftrightarrow \sum_{1\le i<j\le n}\left(1+\frac{a_ia_j}{n-a_ia_j}\right) \le \frac{n^2}{2} \Leftrightarrow \frac{n(n-1)}{2}+\sum_{1\le i<j\le n}\frac{a_ia_j}{n-a_ia_j} \le \frac{n^2}{2}$

$\Leftrightarrow \sum_{1\le i<j\le n}\frac{a_ia_j}{n-a_ia_j} \le \frac{n}{2}$ ①

若存在一次 $a_i^2=n$，则对于 i，$j\in\{1,2\dots n\}$ 均有 $a_j=0$

∴ ① 左 $=0 \le \frac{n}{2}$

若对所有 $i=1,2\cdots n$，均有 $a_i^2<n$，则

$$a_ia_j \le \left(\frac{a_i+a_j}{2}\right)^2 \le \frac{a_i^2+a_j^2}{2}.$$

均值不等式

$$\therefore \frac{a_ia_j}{n-a_ia_j} \le \frac{\left(\frac{a_i+a_j}{2}\right)^2}{n-\frac{a_i^2+a_j^2}{2}} = \frac{(a_i+a_j)^2}{2(\underbrace{n-a_i^2}_{>0}+\underbrace{n-a_j^2}_{>0})} \quad ②$$

由柯西不等式可得

$$\left(\frac{a_j^2}{n-a_i^2}+\frac{a_i^2}{n-a_j^2}\right)(n-a_i^2+n-a_j^2) \ge (a_j+a_i)^2$$

$$\Rightarrow \frac{(a_i+a_j)^2}{n-a_i^2+n-a_j^2} \le \frac{a_j^2}{n-a_i^2}+\frac{a_i^2}{n-a_j^2} \quad ③$$

由②③可得 $\frac{a_ia_j}{n-a_ia_j} \le \frac{1}{2}\left(\frac{a_j^2}{n-a_i^2}+\frac{a_i^2}{n-a_j^2}\right)$

$$\therefore \sum_{1\le i<j\le n} \frac{a_ia_j}{n-a_ia_j} \le \frac{1}{2}\sum_{1\le i<j\le n}\left(\frac{a_j^2}{n-a_i^2}+\frac{a_i^2}{n-a_j^2}\right) = \frac{1}{2}n.$$

27. $1+2abc \geq a^2+b^2+c^2 \iff 1-b^2-c^2 \geq a^2-2bc\cdot a$

$\iff (1-b^2)(1-c^2) \geq (a-bc)^2$ ①

同理，要证不等式 $\iff (1-b^{2n})(1-c^{2n}) \geq (a^n-b^nc^n)^2$

$\iff (1-b^2)\cdot\sum_{i=0}^{n-1} b^{2i}\cdot(1-c^2)\cdot\sum_{i=0}^{n-1}c^{2i} \geq (a-bc)^2\left(\sum_{i=0}^{n-1} a^{n-1-i}b^ic^i\right)^2$ ②

由柯西不等式

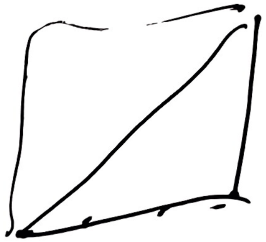
$\sum_{i=0}^{n-1}b^{2i}\cdot\sum_{i=0}^{n-1}c^{2i} \geq \left(\sum_{i=0}^{n-1}|b|^i|c|^i\right)^2 \geq \left(\sum_{i=0}^{n-1}|a|^{n-1-i}|b|^i|c|^i\right)^2$

$\geq \left(\sum_{i=0}^{n-1}a^{n-1-i}b^ic^i\right)^2$ ③

①×③ 可得 ② 式立

28 设 $C=\sum_{j=1}^{n}\frac{1}{c_j}$ 由柯西不等式可得 $\sum_{j=1}^{n}\frac{x_j^2}{a_{ji}}\cdot\sum_{j=1}^{n}a_{ji} \geq \left(\sum_{j=1}^{n}x_j\right)^2$

即 $\sum_{j=1}^{n} \frac{x_j^2}{a_{ji}} \geq \frac{\left(\sum_{j=1}^{n} x_j\right)^2}{\sum_{j=1}^{n} a_{ji}}$ ①



在①中取 $x_j = \frac{1}{c_j}$ 则 $\sum_{j=1}^{n} \frac{\frac{1}{c_j^2}}{a_{ji}} \geq \frac{\left(\sum_{j=1}^{n} \frac{1}{c_j}\right)^2}{\sum_{j=1}^{n} a_{ji}}$

$\Rightarrow \sum_{j=1}^{n} \frac{a_{ij}}{c_j^2} \geq \frac{c^2}{c_i}$ $(i=1,2,\cdots,n)$ ②

对于② 中 $i=1,2\cdots n$ 求和.

$\sum_{i=1}^{n}\sum_{j=1}^{n} \frac{a_{ij}}{c_j^2} \geq c^2 \cdot \sum_{i=1}^{n} \frac{1}{c_i} = c^2 \cdot c = c^3.$

$\therefore \sum_{j=1}^{n}\sum_{i=1}^{n} \frac{a_{ij}}{c_j^2} \geq c^3 \Rightarrow \sum_{j=1}^{n} \frac{1}{c_j^2} \cdot \sum_{i=1}^{n} a_{ij} \geq c^3$

$\Rightarrow \sum_{j=1}^{n} \frac{1}{c_j^2} \cdot c_j \geq c^3 \Rightarrow \sum_{j=1}^{n} \frac{1}{c_j} \geq c^3 \Rightarrow c \geq c^3 \Rightarrow c^2 \leq 1$

$\Rightarrow c \leq 1.$

设 $a=\frac{x-y}{xy+2y+1}$， $b=\frac{y-z}{yz+2z+1}$. $c=\frac{z-x}{zx+2x+1}$

$\frac{1}{a}+1=\frac{xy+2y+1}{x-y}+1=\frac{(x+1)(y+1)}{x-y}=\frac{(x+1)(y+1)}{(x+1)-(y+1)}$

$\Rightarrow \frac{a}{a+1}=\frac{1}{y+1}-\frac{1}{x+1}$. 同理 $\frac{b}{b+1}=\frac{1}{z+1}-\frac{1}{y+1}$. $\frac{c}{c+1}=\frac{1}{x+1}-\frac{1}{z+1}$

$\therefore \frac{a}{a+1}+\frac{b}{b+1}+\frac{c}{c+1}=0 \Rightarrow 1-\frac{1}{a+1}+1-\frac{1}{b+1}+1-\frac{1}{c+1}=0$

$\therefore \frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}=3$. $\because 0<\frac{1}{x+1}, \frac{1}{y+1}\le 1$

$\therefore -1<\frac{a}{a+1}=\frac{1}{y+1}-\frac{1}{x+1}<1$

$-1<1-\frac{1}{a+1}<1$ $\quad 0<\frac{1}{a+1}<2 \Rightarrow a+1>0$

同理 $b+1>0$. $c+1>0$. 由柯西不等式可得

$\left((a+1)+(b+1)+(c+1)\right)\cdot\left(\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}\right)\ge 3^2$

$\Rightarrow (a+b+c+3)\cdot 3\ge 3^2 \Rightarrow a+b+c\ge 0$

30 设 $x=\frac{1}{a}, y=\frac{1}{b}, z=\frac{1}{c}$. 则 $xy+yz+zx=1$

原不等式左 $=\sum\frac{\frac{1}{x}}{\frac{1}{x^2}+1}=\sum\frac{x}{x^2+1}=\sum\frac{x}{x^2+xy+yz+zx}=\sum\frac{x}{(x+y)(x+z)}$

$=\sum\frac{x(y+z)}{(x+y)(y+z)(z+x)}=\frac{\sum xy+\sum xz}{(x+y)(y+z)(z+x)}=\frac{2}{(x+y)(y+z)(z+x)}$

由排序不等式可得 $a^3+b^3\ge a^2b+b^2a=ab(a+b)$

$\frac{\sqrt{a^3+b^3}}{ab+1}\ge\frac{\sqrt{ab(a+b)}}{ab+1}=\frac{\sqrt{\frac{1}{b}+\frac{1}{a}}}{1+\frac{1}{ab}}=\frac{\sqrt{x+y}}{1+xy}$

亦 $\frac{\sqrt{x+y}}{\sqrt{(1+x^2)(1+y^2)}}=\frac{\sqrt{x+y}}{\sqrt{(xy+yz+zx+x^2)(xy+yz+zx+y^2)}}$

均值不等式

$=\frac{\sqrt{\cancel{x+y}}}{\sqrt{(x+y)(x+z)\cancel{(x+y)}(y+z)}}=\frac{1}{\sqrt{(x+y)(y+z)(z+x)}}$ 同理 " "

原不等式左 $\ge\frac{1}{\cancel{3}\sqrt{2}\cdot\sqrt{xyz}}\cdot\frac{\cancel{3}}{\sqrt{(x+y)(y+z)(z+x)}}$ $\frac{2}{(x+y)(y+z)(z+x)}$

$$\Uparrow$$

$$(x+y)^2(y+z)^2(z+x)^2 \geq 8xyz(x+y)(y+z)(z+x)$$

$$\Uparrow$$

$$(x+y)(y+z)(z+x) \geq 8xyz$$

$$\Uparrow$$

$x+y \geq 2\sqrt{xy}$. $y+z \geq 2\sqrt{yz}$. $z+x \geq 2\sqrt{zx}$ 相乘

3\) 先证明 当$n \geq 2$时

$$\frac{x^{n+1}}{x+y^3}+\frac{y^{n+1}}{y+x^3} \geq \frac{1}{2}\left(\frac{x^n}{x+y^3}+\frac{y^n}{y+x^3}\right) \quad ①$$

, 其中$x+y=1$ 而:

$$① \Leftrightarrow \frac{2x^{n+1}-x^n}{x+y^3}+\frac{2y^{n+1}-y^n}{y+x^3} \geq 0$$

$$\Leftrightarrow \frac{2x^{n+1}-x^n(x+y)}{x+y^3}+\frac{2y^{n+1}-y^n(x+y)}{y+x^3} \geq 0$$

$$\Leftrightarrow \frac{x^n(x-y)}{x+y^3}+\frac{y^n(y-x)}{\cancel{y}+\cancel{x^3}} \geq 0$$

$$\Leftrightarrow (x-y)\left(x^n(\cancel{y}+\cancel{x}^3)-y^n(x+y^3)\right) \geq 0$$

$\Leftarrow (x-y)\left(x^{n+3}-y^{n+3}+xy(x^{n-1}-y^{n-1})\right) \geq 0$ ②

$x-y,\ x^{n+3}-y^{n+3},\ x^{n-1}-y^{n-1}$ 均同号. ②得证

因为由① 可得

$$\frac{x^n}{x+y^3}+\frac{y^n}{y+x^3} \geq \frac{1}{2^{n-2}}\left(\frac{x^2}{x+y^3}+\frac{y^2}{y+x^3}\right) \geq \frac{2^{4-n}}{5}$$

$$\Updownarrow$$

$$\frac{x^2}{x+y^3}+\frac{y^2}{y+x^3} \geq \frac{4}{5} \quad ③$$

$$= \frac{xy(x+y)+x^5+y^5}{xy+(xy)^3+x^4+y^4}$$

设 $0<xy=t \leq \left(\frac{x+y}{2}\right)^2=\frac{1}{4}$

$\because x^2+y^2=(x+y)^2-2xy=1-2t \quad x^3+y^3=(x+y)^3-3xy(x+y)=1-3t$

$x^4+y^4=(x+y)^4-4xy(x^2+y^2)-6x^2y^2=1-4t(1-2t)-6t^2=1-4t+2t^2$

$x^5+y^5=(x+y)^5-5xy(x^3+y^3)-10x^2y^2(x+y)=1-5t(1-3t)-10t^2=1-5t+5t^2$

$f(x)=a_nx^n+\cdots+a_1x+a_0$ 是整系数多项式，则其所有有理根形如 $\frac{u}{v}$，$u\in\mathbb{Z}$，$v\in\mathbb{Z}_+$，$(u,v)=1$ 且 $u|a_0$，$v|a_n$

若 $f(\frac{u}{v})=0$ 则 $f(x)=(vx-u)g(x)$，其中 $g(x)$ 是 $n-1$ 次整系数多项式

代入③可得 $\underbrace{(4t-1)}_{\le 0}\underbrace{(t^2-4t+1)}_{\ge 0}\le 0$. ✓

32. a, b, c 互不相等，设 $S_n=\sum\frac{a^n}{(a-b)(a-c)}$ $(n=0,1,\cdots)$

设 $p=a+b+c$, $q=ab+bc+ca$, $r=abc$ 则

$\underbrace{a,b,c}_{\text{特征根}}$ 是三次方程 $x^3-px^2+qx-r=0$ 的三个不同的根
(特征方程)

$\therefore S_{n+3}=pS_{n+2}-qS_{n+1}+rS_n$ $(n=0,1,\cdots)$

$\because S_0=0$, $S_1=0$, $S_2=1$

$\therefore S_3=pS_2-qS_1+rS_0=p\cdot1-q\cdot0+r\cdot0=p$

$S_4=pS_3-qS_2+rS_1=p\cdot p-q\cdot1+r\cdot0=p^2-q$

$S_5=pS_4-qS_3+rS_2=p(p^2-q)-q\cdot p+r\cdot1=p^3-2pq+r$

$S_6=pS_5-qS_4+rS_3=p(p^3-2pq+r)-q(p^2-q)+r\cdot p=p^4-3p^2q+2pr+q^2$

∵ $r=1$ ∴ $S_6 = p^2(p^2-3q) + 2p + q^2$

∵ $p^2-3q = (a+b+c)^2 - 3(ab+bc+ca) > 0$ ∴ $p^2(p^2-3q) > 0$

需确定.

由均值不等式可得 $2p = 2(a+b+c) > 2\cdot 3\sqrt[3]{abc} = 6$

$q^2 = (ab+bc+ca)^2 > \left(3\sqrt[3]{ab\cdot bc\cdot ca}\right)^2 = 9$.

∴ $S_6 > 0+6+9 = 15$.

33 $abc\left(\sum x\right)^2 \geq \sum ayz(1-2a) \iff abc\left(\sum x\right)^2 \geq \sum ayz\left((a+b+c)^2 - 2a(a+b+c)\right)$

$\iff abc(x^2+y^2+z^2+2xy+2yz+2zx) \geq \sum ayz(b^2+c^2-a^2+2bc)$

$\iff abc(x^2+y^2+z^2) \geq \sum ayz(b^2+c^2-a^2) \iff x^2+y^2+z^2 \geq \sum 2yz\cdot\frac{b^2+c^2-a^2}{2bc}$ ①

∵ $b+c = 1-a > 1-\frac{1}{2} = \frac{1}{2} > a$ 同理 " "

∴ 存在 $\triangle ABC$ 使得 $BC=a$. $CA=b$. $AB=c$. 由余弦定理

可得 $\cos A = \frac{b^2+c^2-a^2}{2bc}$ " " "

① 化为 $x^2+y^2+z^2 \geq 2(yz\cos A+zx\cos B+xy\cos C)$ 嵌入不等式

35 (1) 由 $f(1)=1$ 可得 $1=\frac{2\cdot 1}{a\cdot 1+b} \Rightarrow a+b=2$.

由 $f(\frac{1}{2})=\frac{2}{3}$ 可得 $\frac{2}{3}=\frac{2\cdot\frac{1}{2}}{a\cdot\frac{1}{2}+b}=\frac{2}{a+2b} \Rightarrow a+2b=3$. $\Rightarrow a=b=1$

$\therefore f(x)=\frac{2x}{x+1}$ $x_{n+1}=f(x_n)=\frac{2x_n}{x_n+1}$ $(n\in N^*)$ 由 $x_1=\frac{1}{2}$

方法一. $\frac{1}{x_{n+1}}=\frac{1}{2}(1+\frac{1}{x_n})$

方法二. $x_2=\frac{2}{3}$ $x_3=\frac{4}{5}$ $x_4=\frac{8}{9}$ 下面证明 $x_n=\frac{2^{n-1}}{2^{n-1}+1}$ ①

假设①对 n 成立. 则

$$x_{n+1}=\frac{2\cdot\frac{2^{n-1}}{2^{n-1}+1}}{\frac{2^{n-1}}{2^{n-1}+1}+1}=\frac{2^n}{2^n+1}$$ ①成立.

(2) $(1+\frac{1}{n})^n<e<(1+\frac{1}{n})^{n+1}$

原不等式 $\Leftrightarrow \dfrac{1}{x_1x_2\cdots x_{n+1}} < 2e$ ②

② $\Leftrightarrow 2\cdot(1+\frac{1}{2})(1+\frac{1}{2^2})\cdots\cdots(1+\frac{1}{2^n}) < 2e$ ③

伯努利不等式. $x>-1$

① 若 $\alpha\le 0$ 或 $\alpha\ge 1$, 则 $(1+x)^\alpha \ge 1+\alpha x$

② 若 $0\le\alpha\le 1$, 则 $(1+x)^\alpha \le 1+\alpha x$

$x>0$ $\quad (1+x)^n \ge 1+nx$.

$n\in\mathbb{Z}_+$

由伯努利不等式可得

$$③左 = 2(1+2^{n-1}\cdot\frac{1}{2^n})(1+2^{n-2}\cdot\frac{1}{2^n})\cdots(1+2\cdot\frac{1}{2^n})(1+\frac{1}{2^n})$$

$$\le 2(1+\frac{1}{2^n})^{2^{n-1}}\cdot(1+\frac{1}{2^n})^{2^{n-2}}\cdots(1+\frac{1}{2^n})^2(1+\frac{1}{2^n})$$

$$= 2(1+\frac{1}{2^n})^{2^{n-1}+2^{n-2}+\cdots+2+1} = 2(1+\frac{1}{2^n})^{2^n-1} < 2(1+\frac{1}{2^n})^{2^n} < 2e$$

证毕