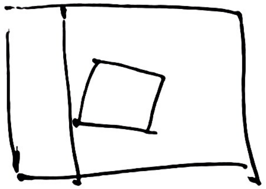
统 ~ ㊀



$$\frac{0}{0}=1$$

$$\Leftrightarrow 0=0\times 1$$

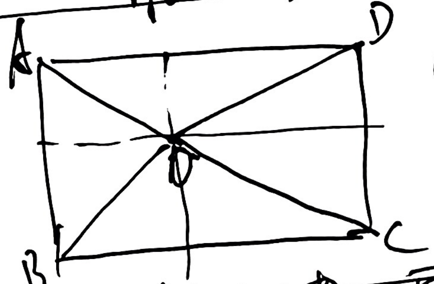
㊁ 向量法

㊂ 特殊化策略

同方法

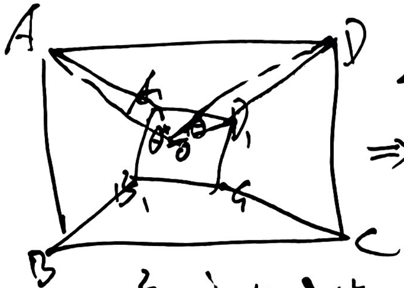
同结果

缩为一个点



$OA^2+OC^2=OB^2+OD^2$

该结论能否解决一般情况？



$\triangle OAD \sim \triangle OA_1D_1$

$\Rightarrow \triangle OAA_1 \sim \triangle ODD_1$

极端化

$AA_1^2 = OA^2 + OA_1^2 - 2OA\cdot OA_1\cos\theta$

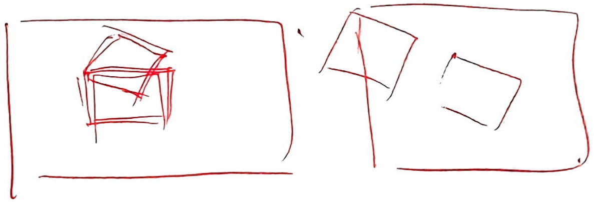
$BB_1^2 = OB^2 + OB_1^2 - 2OB\cdot OB_1\cos\theta$

$CC_1^2 = OC^2 + OC_1^2 - 2OC\cdot OC_1\cos\theta$

$DD_1^2 = OD^2 + OD_1^2 - 2OD\cdot OD_1\cos\theta$

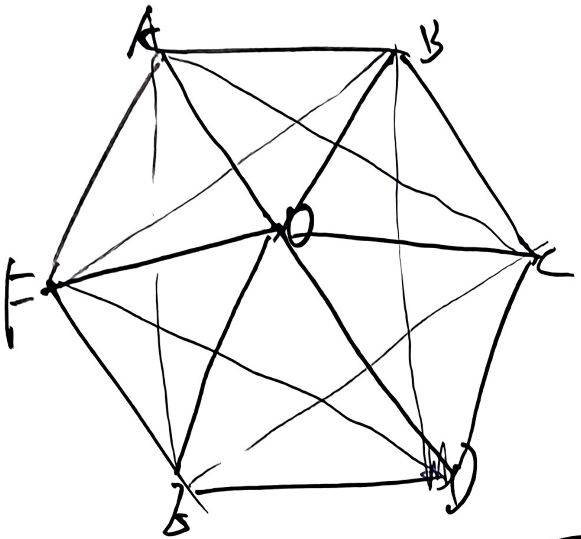
kOA^2 kOB^2 kOC^2 kOD^2

一般化



小范围缩为个点

$$OA^2+OC^2+OE^2=OB^2+OD^2+OF^2$$

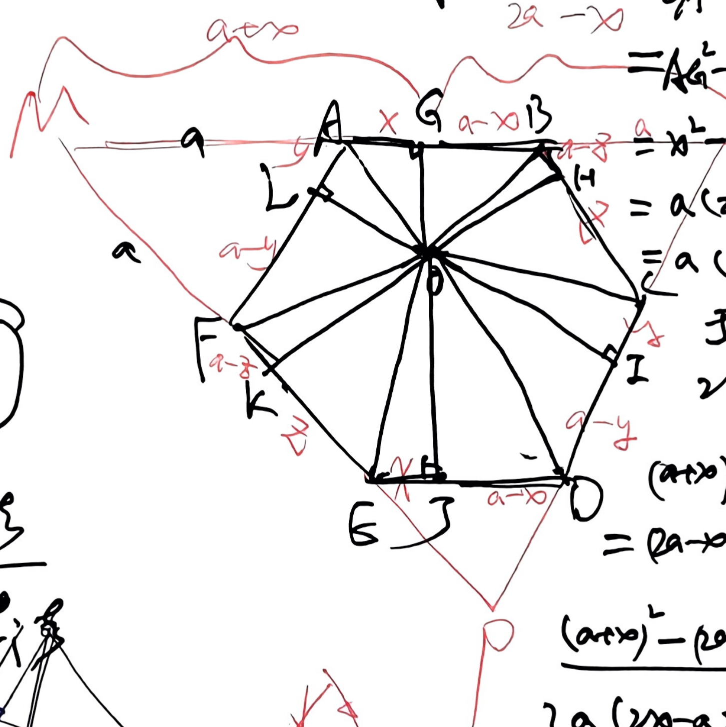


$$OA^2+OD^2=OB^2+OE^2=OC^2+OF^2 \quad ?$$

$$OA^2-OB^2+OC^2-OD^2+OE^2-OF^2$$
$$=AG^2-BG^2+CI^2-DI^2+EK^2-FK^2$$
$$=x^2-(a-x)^2+\cdots+\cdots$$
$$=a(2x-a)+\cdots+\cdots$$
$$=a(2x+2y+2z-3a)$$

于是只需证

$$2x+2y+2z=3a \quad ?$$



$$(a+x)^2+(a+y)^2+(a+z)^2=(2a-x)^2+(2a-y)^2+(2a-z)^2$$

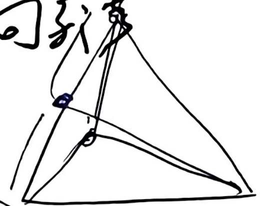
$$(a+x)^2-(2a-x)^2+(a+y)^2-(\ )+\cdots=0$$

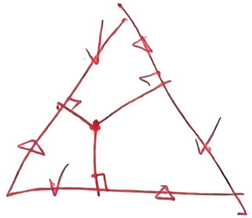
$$3a(2x-a)+\cdots+\cdots=0$$

$$3a(2x-a+2y-a+2z-a)=0$$

$$\therefore 2x+2y+2z=3a$$

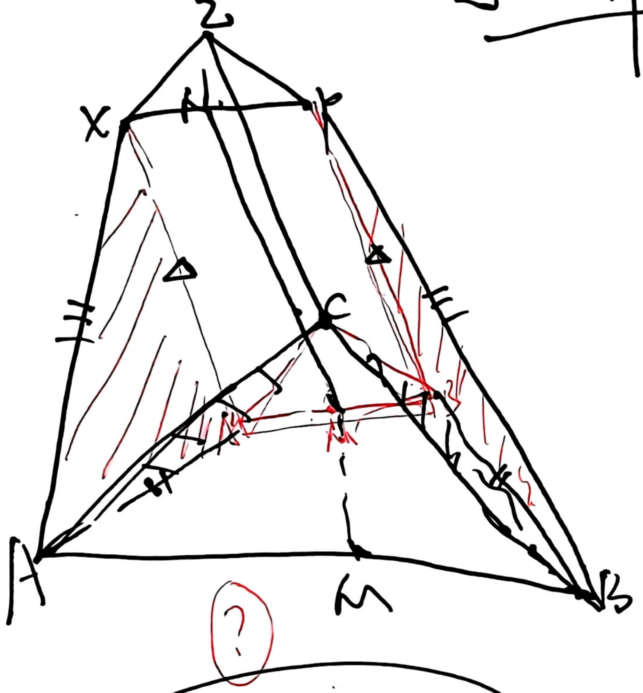
向量





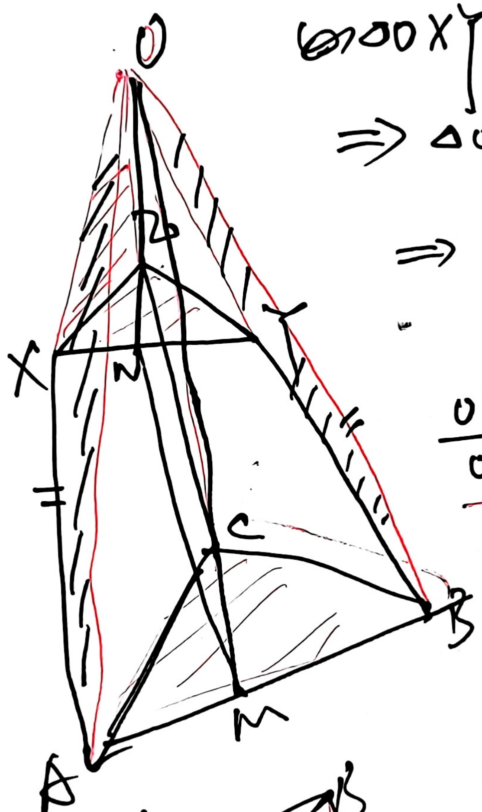
如意算盘

中线对称α



作截面的中心S

尝试解读



由$\triangle OXY \sim \triangle OAB$

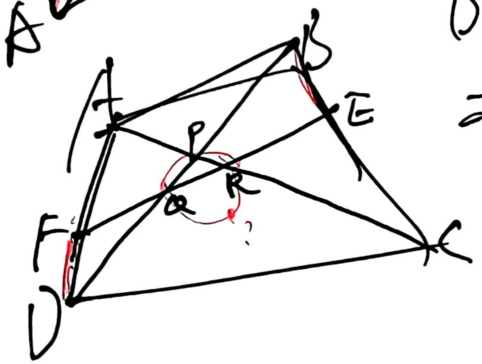
$\Rightarrow \triangle OXA \cong \triangle OYB$

$\Rightarrow OA = OB$

$OX = OY$

$\dfrac{OZ}{OC} = \dfrac{XZ}{AC} = \dfrac{ZN}{CM}$

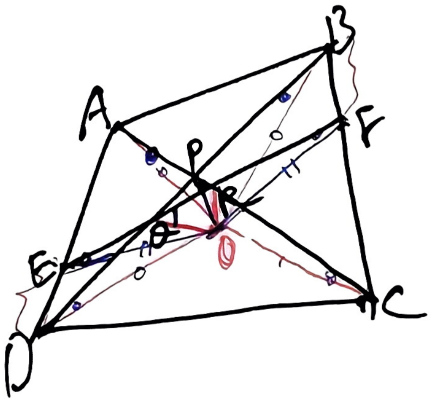
中



$BC = AD$, $BE = DF$

求证 $\triangle PQR$过定点

(引非圆)



AED 与 CFB 相似

~~△OAD ∽ △OCB~~

△DOA ≌ △BOC

△AOC. △BOD △EOF

为相似的剪像.

∠OPR = ∠OPC = ∠OBC

⇑

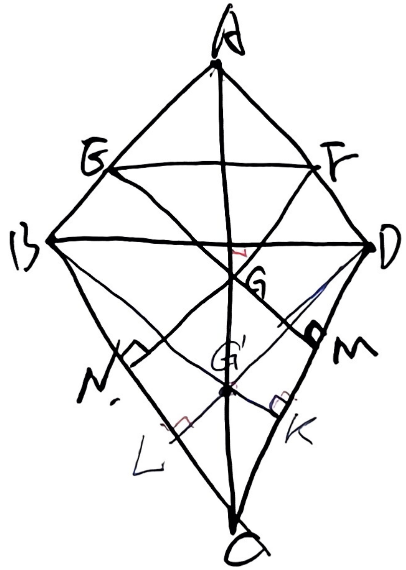
P.O.C.B 共圆

∠OQR = ∠EDO = ∠ODA

⇑

Q.E.D.O 共圆

97



作 $BK \perp CD$ 于 K 交 AC 于 G'，连 DG' 并延长交 BC 于 L.

$BK \parallel EM \Rightarrow \frac{AE}{EB} = \frac{AG}{GG'}$

$EF \parallel BD \Rightarrow \frac{AE}{GB} = \frac{AF}{FD}$

$\Rightarrow \frac{AG}{GG'} = \frac{AF}{FD}$

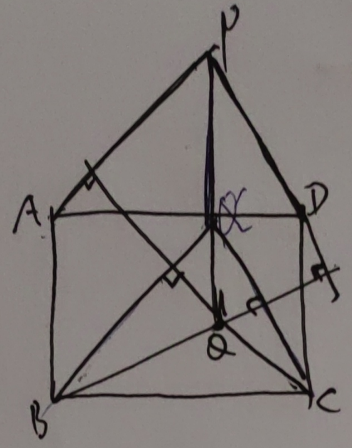
$\Rightarrow FG \parallel DG'$

即 $FN \parallel DL$，$FN \perp BC$ $\Rightarrow DL \perp BC$

$\therefore G'$ 为 $\triangle BCD$ 垂心

$\therefore AC \perp BD$

与



作 $PQ' \perp AB$ } ⟹ $PQ' \perp BC$

[illegible]

Q 为 $\triangle Q'BC$ 的垂心

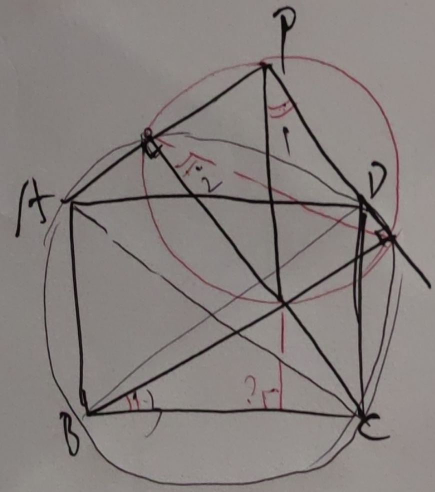
$\therefore QQ' \perp BC$

过 Q' ~~有且只有一条~~ 直线

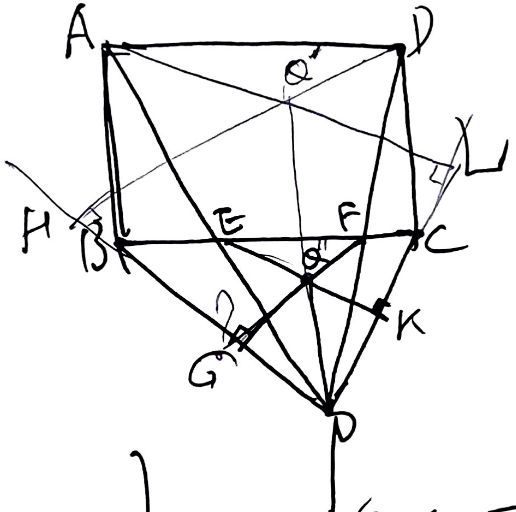
与 BC 垂直

$\therefore P、Q'、Q$ 三点共线

$\therefore PQ \perp BC$



$\angle 1 = \angle 2 = \angle 3$



刚才结论能利用吗：

$PQ' \perp BC$?

下证 P, Q, Q' 共线

变通？

$FG \parallel DH \iff \dfrac{PQ''}{Q'Q''} = \dfrac{PG}{HG}$ (?)

$Q''K \parallel Q'L \iff \dfrac{PQ''}{Q'Q''} = \dfrac{PK}{KL}$ ✓

只需证 $\dfrac{PG}{HG} = \dfrac{PK}{KL}$

$FG \parallel DH \Rightarrow \dfrac{PG}{HG} = \dfrac{PF}{DF}$

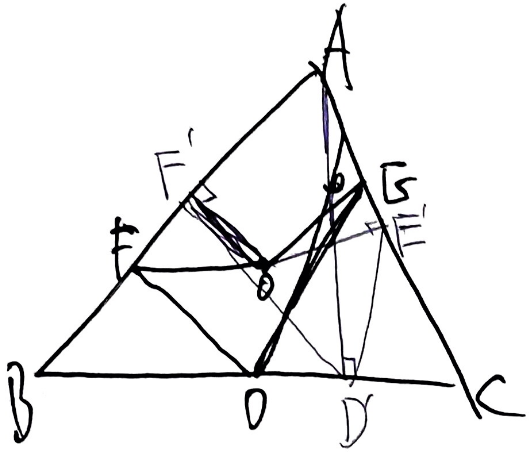
$EK \parallel AL \Rightarrow \dfrac{PK}{KL} = \dfrac{PE}{AE}$

只需证 $\dfrac{PF}{DF} = \dfrac{PE}{AE}$ ✓

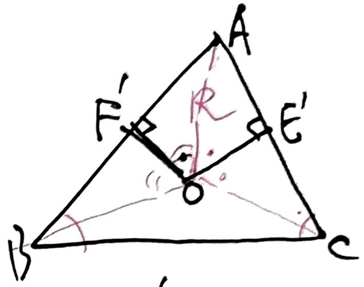
$\Uparrow$

$EF \parallel AD$

例3.



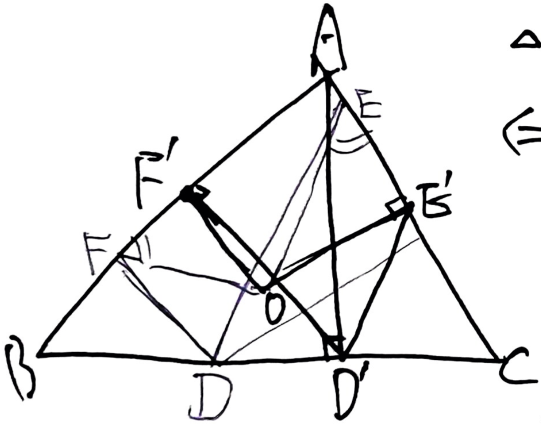
$$= \frac{\cos B}{\cos C}$$



$$\frac{OE'}{OF'} = \frac{OA \cdot \cos\angle AOE'}{OA \cos\angle AOF'}$$

$$= \frac{\cos B}{\cos C}$$

三弦定理 ?



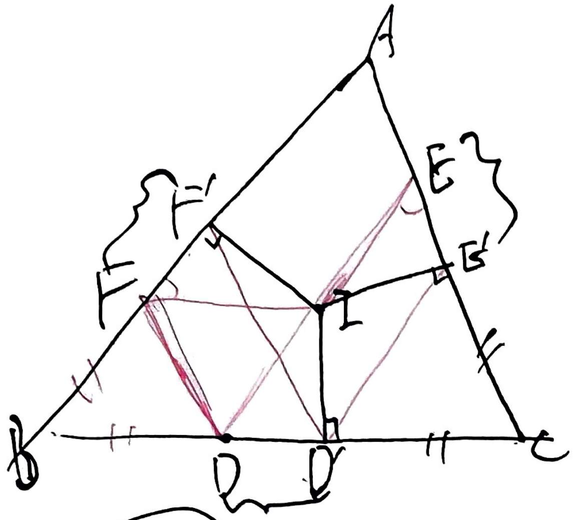
$\triangle OE'E \sim \triangle OF'F$

$$\Leftarrow \frac{EE'}{FF'} = \frac{OE'}{OF'}$$

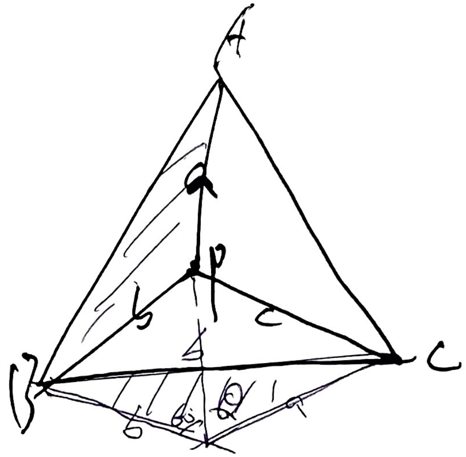
$$\frac{EE'}{FF'} = \frac{EE'}{DD'} \cdot \frac{DD'}{FF'}$$

$$= \frac{OE'}{OD'} \cdot \frac{BD'}{BF'}$$

$$= \frac{\frac{1}{2}AC}{AC\cos C} \cdot \frac{AB\cos B}{\frac{1}{2}AB}$$



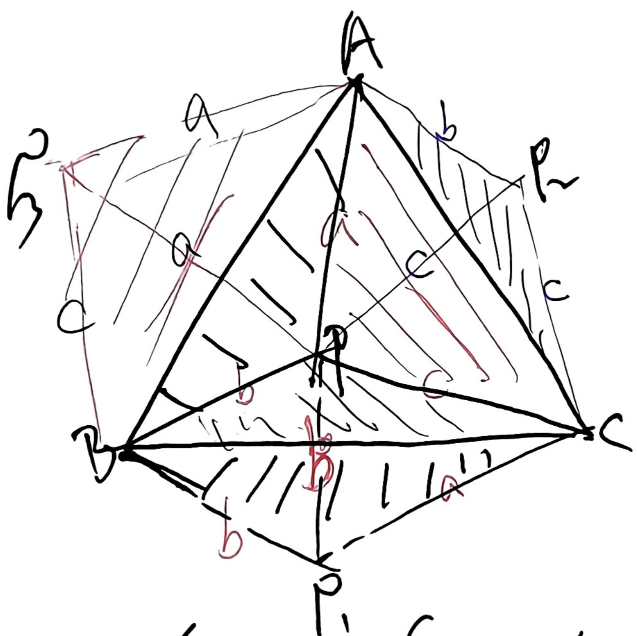
$$\frac{EE'}{FF'} = \frac{IE'}{IF'} = 1$$

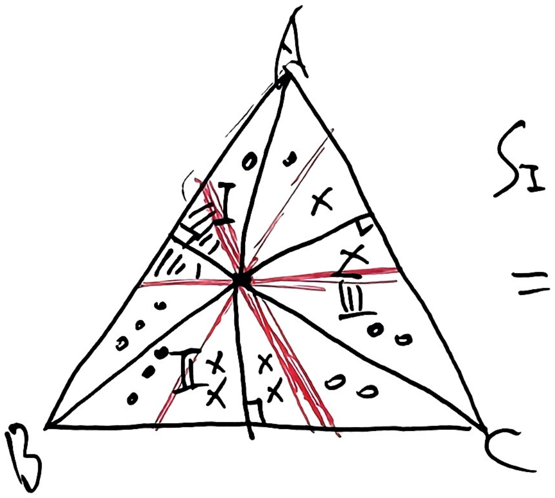


$$\frac{\sqrt{3}}{4}BC^2 = \frac{\sqrt{3}}{4}\left(a^2+b^2-2ab\underbrace{\cos(60^\circ+\theta)}_{\left(\frac{1}{2}\cos\theta-\frac{\sqrt{3}}{2}\sin\theta\right)}\right)$$

$\sqrt{3}$

$$\cos\theta = \frac{a^2+b^2-c^2}{2ab} \qquad \sin\theta = ?$$





$$S_{\mathrm{I}} + S_{\mathrm{II}} + S_{\mathrm{III}} = \frac{1}{2} S_{\triangle ABC}$$

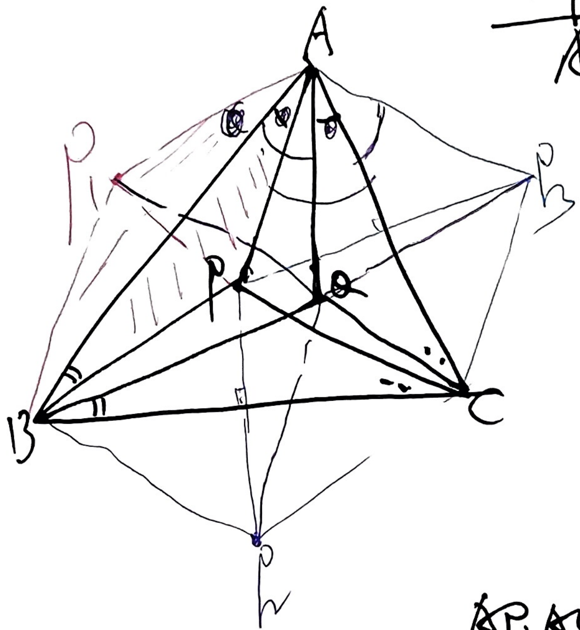
$$2S_{\triangle ABC} = S_{六边形}$$

$$= \frac{\sqrt{3}}{4}(a^2+b^2+c^2) + 3\sqrt{p(p-a)(p-b)(p-c)}$$

$$S_{\triangle ABC} = \frac{\sqrt{3}}{8}(a^2+b^2+c^2) + \frac{3}{2}\sqrt{p(p-a)(p-b)(p-c)}$$

（其中 $p=\frac{a+b+c}{2}$）

$$\frac{AP\cdot AQ}{AB\cdot AC}+\frac{BP\cdot BQ}{AB\cdot BC}+\frac{CP\cdot CQ}{BC\cdot AC}=1$$



$$S_{六边形}=2S_{\triangle ABC}$$

$$=S_{\triangle QAP_1}+S_{\triangle QAP_3}+\cdots+\cdots+\cdots+\cdots$$

$$=\frac{1}{2}\cdot AP_1\cdot AQ\sin\angle P_1AQ+\frac{1}{2}AQ\cdot AP_3\sin\angle P_3AQ+\cdots$$

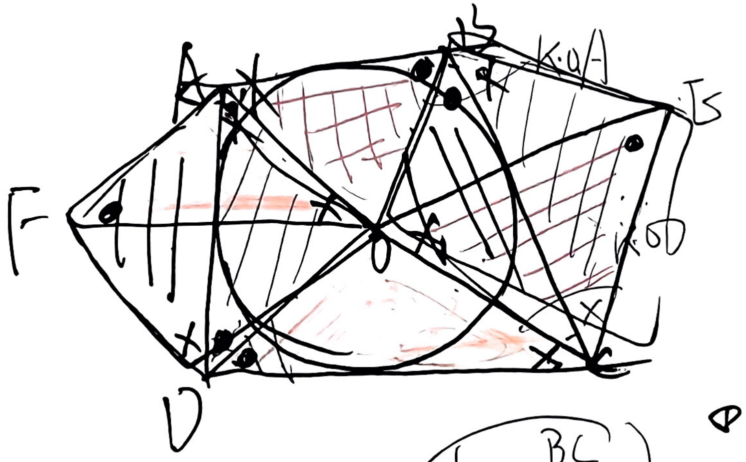
$$=\frac{1}{2}AP\cdot AQ\sin A+\frac{1}{2}AQ\cdot AP\sin A+\cdots$$

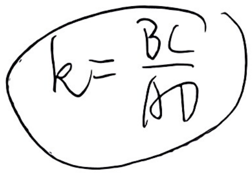
$$=AP\cdot AQ\sin A+BP\cdot BQ\sin B+CP\cdot CQ\sin C$$

$$1=\frac{AP\cdot AQ\sin A}{2S_{\triangle ABC}}+\frac{BP\cdot BQ\sin B}{2S_{\triangle ABC}}+\frac{CP\cdot CQ\sin C}{2S_{\triangle ABC}}$$

$$=\frac{AP\cdot AQ\cancel{\sin A}}{AB\cdot AC\cancel{\sin A}}+\frac{BP\cdot BQ\cancel{\sin B}}{AB\cdot BC\cancel{\sin B}}+\frac{CP\cdot CQ\cancel{\sin C}}{BC\cdot AC\cancel{\sin C}}$$

$$OA\cdot OC+OB\cdot OD=\sqrt{abcd}$$





$$OB\cdot CE+OC\cdot BE=BC\cdot OE$$

$$OB\cdot\frac{BC}{AD}\cdot OD+OC\cdot\frac{BC}{AD}\cdot OA=BC\cdot OE$$

$$OB\cdot OD+OC\cdot OA=AD\cdot OE \quad ①$$

$$OB\cdot OD+OC\cdot OA=BC\cdot OF \quad ②$$

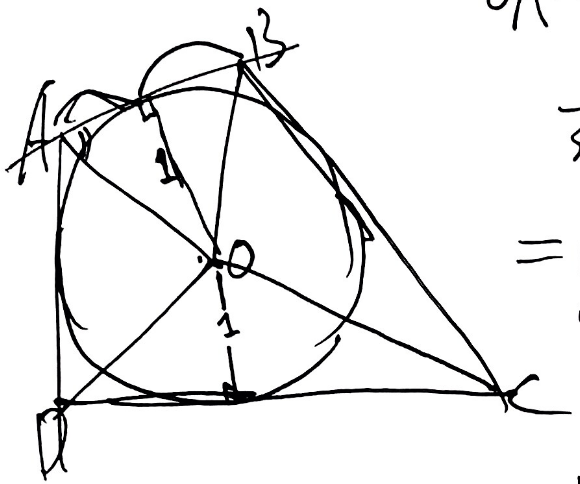
①×②

$$\Rightarrow OA\cdot OC+OB\cdot OD=\sqrt{AD\cdot BC\cdot OE\cdot OF}$$

只需证 $OE\cdot OF=AB\cdot CD$

$$\frac{AB}{OE}=\frac{OA}{OC} \qquad \frac{CD}{OF}=\frac{OC}{OA}$$

$$\frac{AB}{OE}\cdot\frac{CD}{OF}=1 \quad \therefore OE\cdot OF=AB\cdot CD$$



$OA\cdot OC + OB\cdot OD = \sqrt{abcd}$

$\frac{1}{\sin\frac{A}{2}}\cdot\frac{1}{\sin\frac{C}{2}} + \frac{1}{\sin\frac{B}{2}}\cdot\frac{1}{\sin\frac{D}{2}}$

$= \sqrt{\left(\cot\frac{A}{2}+\cot\frac{B}{2}\right)\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)\left(\cot\frac{C}{2}+\cot\frac{D}{2}\right)\left(\cot\frac{D}{2}+\cot\frac{A}{2}\right)}$

$\Leftrightarrow \frac{\sin\frac{A}{2}\sin\frac{C}{2}+\sin\frac{B}{2}\sin\frac{D}{2}}{\cancel{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2}}} = \frac{\sqrt{\sin\frac{A+B}{2}\sin\frac{B+C}{2}\sin\frac{C+D}{2}\sin\frac{D+A}{2}}}{\cancel{\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin\frac{D}{2}}}$ ③

巧

$\cot\frac{A}{2}+\cot\frac{B}{2}$

$= \frac{\cos\frac{A}{2}}{\sin\frac{A}{2}} + \frac{\cos\frac{B}{2}}{\sin\frac{B}{2}}$

$= \frac{\sin\frac{A+B}{2}}{\sin\frac{A}{2}\sin\frac{B}{2}}$

$A+B+C+D = 360$

$\frac{A+B+C+D}{2} = 180$

← 隐含条件

$\Leftrightarrow \sin\frac{A}{2}\sin\frac{C}{2} + \sin\frac{B}{2}\sin\frac{D}{2} = \sin\frac{A+B}{2}\sin\frac{B+C}{2}$

?

$\frac{1}{2}\left(\cos\frac{A-C}{2} - \cos\frac{A+C}{2}\right) + \frac{1}{2}\left(\cos\frac{B-D}{2} - \cos\frac{B+D}{2}\right)$

$360-D$

$= \frac{1}{2}\left(\cos\frac{A-C}{2} - \cos\frac{A+B+B+C}{2}\right)$

$\left(180 - \frac{B-D}{2}\right)$