1、已知正整数n满足，对开区间(0, 2024)内每个整数m，总存在正整数k，使得 $\frac{m}{2024}<\frac{k}{n}<\frac{m+1}{2025}$. 求满足条件的n的最小值.

$$\frac{c}{a}<\frac{d}{b}$$

$$\boxed{\frac{c}{a}<\frac{c+d}{a+b}<\frac{d}{b}}$$

$$\frac{m}{x}<\frac{k}{n}<\frac{m+1}{x+1}$$

m取 $1\sim x-1$. $x=2$, $n=5$.

$$\frac{2}{4}=\frac{1}{2}<\frac{2}{3}<\frac{3}{4}.$$

$$\frac{1}{2}<\frac{3}{5}<\frac{2}{3}.$$

$x=3$

$$\frac{1}{3}<\frac{}{n}<\frac{2}{4}, \quad n=5.$$

$$\frac{3}{5}<\frac{2}{3}<\frac{}{n}<\frac{3}{4}<\frac{4}{5}$$

$$\frac{4}{6}=\boxed{\frac{2}{3}<\frac{}{6}<\frac{3}{4}}<\frac{5}{6}$$

$$\frac{2}{3}<\frac{5}{7}<\frac{3}{4}$$

$x=3$, $n=7$.

$n=2x+1$?

$$\frac{x-1}{x}<\frac{k}{n}<\frac{x}{x+1} \quad (*)$$

$n=2x+1$. 对(*)行 ①

$$\frac{x-1}{x}<\frac{2x-1}{2x+1}<\frac{x}{x+1}$$

$x=4$.

$$\frac{1}{4}<\frac{}{n}<\frac{2}{5}$$

$$\frac{2}{4}<\frac{}{n}<\frac{3}{5}$$

$$\boxed{\frac{3}{4}<\frac{}{n}<\frac{4}{5}}$$

n小

$$\boxed{\frac{n-2}{n}\le\frac{3}{4}<\frac{4}{5}<\frac{n-1}{n}}$$

$$\frac{m}{x}<\frac{k}{n}<\frac{m+1}{x+1}$$

$$\boxed{\frac{m}{x}<\frac{2m+1}{2x+1}<\frac{m+1}{x+1}}$$

$\frac{x-1}{x} < \frac{k}{n} < \frac{x}{x+1}$ (**), 记 $n \le 2x$ 时 不行②.

要 $\boxed{n \le 2x}$!

$\frac{1}{n} \ge \frac{1}{2x}$

(**) $\iff \frac{1}{x} > \frac{n-k}{n} > \frac{1}{x+1}$

$\frac{1}{x} > \frac{r}{n} > \frac{1}{x+1}$.

本质说.

$\boxed{\frac{2}{n} \ge \frac{1}{x}} > \frac{1}{x+1} \ge \frac{1}{n}$ ...

$\frac{1}{x+1} \ge \frac{1}{n}$ ← 结构前说

$\boxed{n \ge x+1}$

$\frac{2}{n} \ge \frac{1}{x} \iff 2x \ge n$.

若 $n \le 2x$, 则

(**) $\iff \frac{1}{n} > \frac{n-k}{n} > \frac{1}{x+1}$.

设 $n-k=r$.

一方面. $n \ge x+1$ (显然).

$\therefore \frac{r}{n} > \frac{1}{x+1} \ge \frac{1}{n}$.

$\Rightarrow r > 1 \Rightarrow r \ge 2$.

但 $\frac{r}{n} \ge \frac{2}{2x} = \frac{1}{x}$

这与 $\frac{1}{x} > \frac{r}{n}$ 矛盾!

$\therefore n \le 2x$ 不合题意.

2、$x,y,z\in\mathbb{R}^+$，且满足 $\begin{cases}\frac{2}{5}\le z\le\min\{x,y\}\\ xz\ge\frac{4}{15}\\ yz\ge\frac{1}{5}\end{cases}$ 求 $\frac{1}{x}+\frac{2}{y}+\frac{3}{z}$ 的最大值.

$\boxed{\frac{1}{\sqrt{x}}\le\frac{\sqrt{15}}{2}\sqrt{z}}$

法一：$\frac{1}{x}+\frac{1}{z}+2\left(\frac{1}{y}+\frac{1}{z}\right)$.

$=\frac{2}{x}+\left(\frac{1}{z}-\frac{1}{x}\right)+2\left[\frac{2}{y}+\frac{1}{z}-\frac{1}{y}\right]$

$=\frac{2}{x}+\frac{1}{z}\left(1-\frac{z}{x}\right)+2\left[\frac{2}{y}+\frac{1}{z}\left(1-\frac{z}{y}\right)\right]$

$\frac{1}{x}+\frac{1}{z}=\frac{2}{x}+\frac{1}{z}\left(1-\frac{z}{x}\right)$

$\le\frac{2}{x}+\frac{5}{2}\left(1-\frac{z}{x}\right)$

$\le\frac{\sqrt{z}}{\sqrt{x}}\cdot\frac{\sqrt{15}}{2}+\frac{5}{2}-\frac{5}{2}\frac{z}{x}$

$=-\frac{5}{2}\left(\boxed{\frac{\sqrt{z}}{\sqrt{x}}}-\frac{\sqrt{15}}{5}\right)^2+4$

≤ 4

法二：$\frac{1}{x}=a,\ \frac{2}{y}=b,\ \frac{3}{z}=c.$

$x=\frac{1}{a},\ y=\frac{2}{b},\ z=\frac{3}{c}.$

$\begin{cases}c\le\frac{15}{2}\\ a\le\frac{1}{3}c\\ b\le\frac{2}{3}c.\\ a\le\frac{45}{4}\cdot\frac{1}{c}\\ b\le 30\cdot\frac{1}{c}\end{cases}$

$f(a,b,c)=a+b+c.$

a 取 $\min\{\frac{1}{3}c,\ \frac{45}{4}\frac{1}{c}\}$.

b 取 $\min\{\frac{2}{3}c,\ 30\cdot\frac{1}{c}\}$.

a 取 $\frac{2}{4}$. x 取 $\frac{2}{3}$.

$$\max\{a,b,c\} = \max_{c\in[0,\frac{15}{2}]} \left\{c+\min\left\{\frac{1}{3}c, \frac{45}{4}\cdot\frac{1}{c}\right\} + \min\left\{\frac{2}{3}c, 30\cdot\frac{1}{c}\right\}\right\}$$

$\frac{1}{3}c = \frac{45}{4}\cdot\frac{1}{c}$ $\qquad$ $\frac{2}{3}c = 30\cdot\frac{1}{c}$

$c^2 = \frac{135}{4}$. $\qquad$ $c^2 = 45$. $\qquad$ $c^2 \le \frac{225}{4}$.

c^2	$0, \frac{135}{4}$	$\frac{135}{4}, 45$	$45, \frac{225}{2}$
a	$\frac{1}{3}c$	$\frac{45}{4}\cdot\frac{1}{c}$	$\frac{45}{4}\cdot\frac{1}{c}$
b.	$\frac{2}{3}c$.	$\frac{2}{3}c$	$30\cdot\frac{1}{c}$

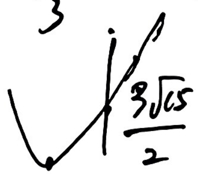
① $c^2 \le \frac{135}{4}$

$f(a,b,c) = 2c \le 2\cdot\sqrt{\frac{135}{4}} = 3\sqrt{15}$.

② $\frac{135}{4} \le c^2 \le 45$.

$f(a,b,c) = \frac{2}{3}c + c + \frac{45}{4}\cdot\frac{1}{c}$

$= \frac{5}{3}c + \frac{45}{4}\cdot\frac{1}{c}$.



$\frac{21}{4}\sqrt{5}$.

③ $c \ge 45$.

$f(a,b,c) = \frac{165}{4}\cdot\frac{1}{c} + c$

c 取 $\frac{15}{2}$.

(13).

$+\sqrt{|x_n - x_1|}$ 的最大值.

此时 z 取 $\frac{2}{5}$. x 取 $\frac{2}{3}$. y 取 $\frac{1}{2}$.

$\frac{\sqrt{z}}{\sqrt{x}}$ 存在

$\frac{\sqrt{15}}{5}$ 此取 max.

$$\left|\frac{1}{x} + \cancel{} \frac{t}{z}\right|$$

$$= \frac{1+t}{x} + \frac{t}{z} - \frac{t}{x}$$

$$= \frac{1+t}{x} + \frac{t}{z}\left(1 - \frac{z}{x}\right)$$

$$= \frac{1+t}{x} + \frac{5t}{2} - \frac{5t}{2} \cdot \frac{z}{x}$$

O7

$$\le \frac{1+t}{\sqrt{x}} \cdot \frac{\sqrt{15}}{2}\sqrt{z} + \frac{5}{2}t - \frac{5}{2}t \cdot \frac{z}{x}$$

$$= -\frac{5}{2}t \cdot \left(\frac{\sqrt{z}}{\sqrt{x}}\right)^2 + \frac{(1+t)\sqrt{15}}{2} \cdot \left(\frac{\sqrt{z}}{\sqrt{x}}\right) + \frac{5}{2}t.$$

$$-\frac{b}{2a} \qquad \frac{(1+t)\sqrt{15}}{2 \times 5t} = \frac{\sqrt{15}}{5} \Rightarrow 5(1+t) = 10t$$

$$t = 1.$$

$$\frac{1}{x} + \frac{2}{y} + \frac{10}{z}.$$

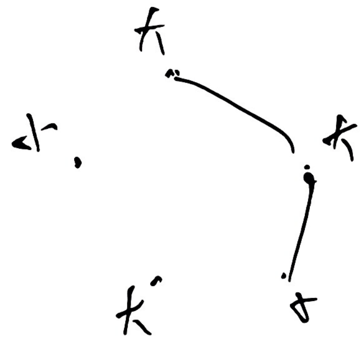
$$\frac{1}{4} + \frac{1}{z} \le 4$$

$$\frac{2}{y} + \frac{2}{z} \le 9.$$

$$\frac{7}{2} \le \frac{5}{2} \times 7$$

3、$n\geq 3$ 为奇数，求 $E=\sqrt{|x_1-x_2|}+\sqrt{|x_2-x_3|}+\cdots+\sqrt{|x_{n-1}-x_n|}+\sqrt{|x_n-x_1|}$ 的最大值.

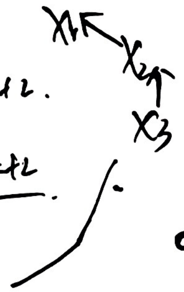
其中 $x_i\in[0,1]$, $i=1,2,\cdots,n$.

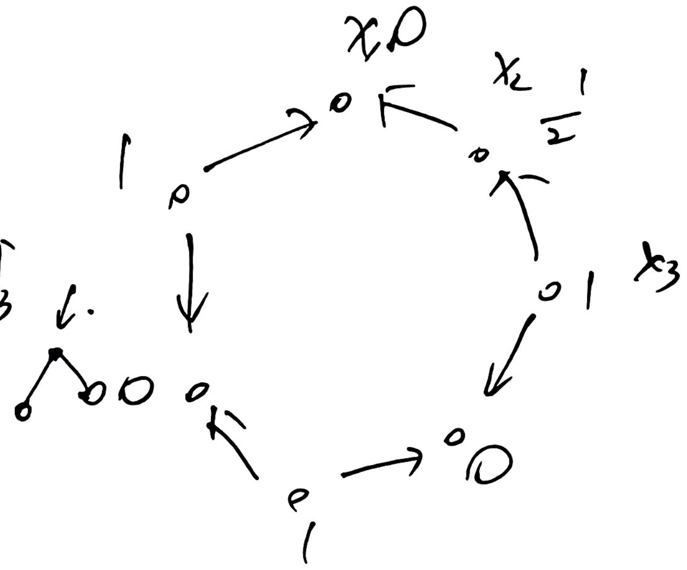


$x_1\leq x_2\leq x_3$.

$x_k\leq x_{k+1}\leq x_{k+2}$.

$x_k\geq x_{k+1}\geq x_{k+2}$.





(1) $\exists$ ~~……~~ $x_i\leq x_{i+1}\leq x_{i+2}$.

(2) 求最值.

不妨设 $x_1\leq x_2\leq x_3$.

(2) $E=\sqrt{|x_1-x_2|}+\sqrt{|x_2-x_3|}+\cdots$

$\leq \cancel{\sqrt{|x_1-x_2|}+\sqrt{|x_2-x_3|}}+n-2$.

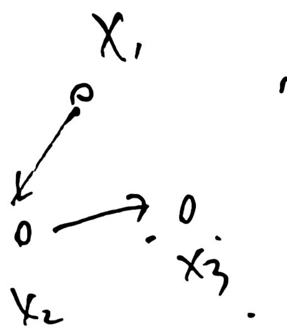
$\sqrt{x_3-x_2}+\sqrt{x_2-x_1}$

$\leq\sqrt{1-x_2}+\sqrt{x_2}+n-2$.

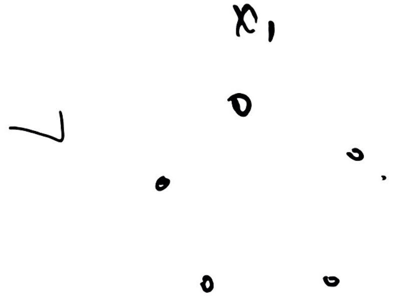
$\leq\sqrt{2}+n-2$ ✓.

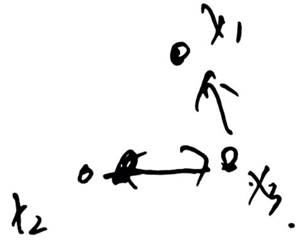
$(\sqrt{1-x_2}+\sqrt{x_2})^2\leq((1-x_2)+x_2)(1+1)$.

$=2$.



$(x_1 \ge x_2 \ge x_3)$

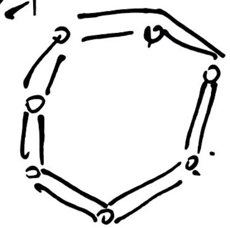




$\cancel{x_2 \ge x_3 \ge x_2}$

$x_2 \le x_3 \le x_1$

$$\prod_{k=1}^{n}(x_{k+2}-x_{k+1})(x_{k+1}-x_k) = \left[\prod_{k=1}^{n}(x_{k+1}-x_k)\right]^2 \ge 0$$



$(x_{k+1}-x_k)(x_{k+2}-x_{k+1}) \ge 0$

$\Updownarrow$

假设不存在 $x_k \le x_{k+1} \le x_{k+2}$ 或 $x_k \ge x_{k+1} \ge x_{k+2}$

不妨设 $x_1 > x_2$

$\Downarrow$

$\cancel{x_2 > x_3}$

$x_2 < x_3$ $\Downarrow$

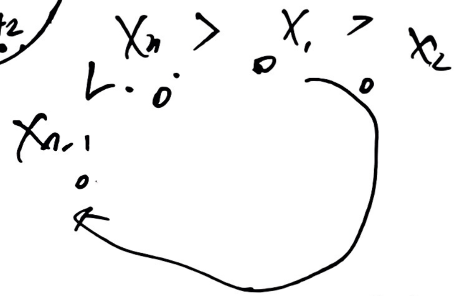
$x_3 > x_4$

$\Downarrow$

$\vdots$

$x_{n-1} < x_n$

$x_n > x_{n+1} = x_1$

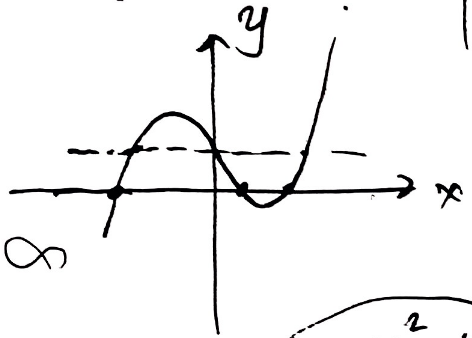


$\therefore x_n > x_1 > x_2$. 矛盾!

4、已知 $\boxed{x^3-3x+1=0}$ 的三根为 a,b,c，且 $a<b<c$，求 a^2+b^2-ac 的值.

$$\begin{cases} a+b+c=0 \\ ab+bc+ca=-3 \\ abc=-1 \end{cases} \Rightarrow a=-b-c$$

$$\begin{cases} bc+(b+c)(-b-c)=-3 \\ (-b-c)bc=-1 \end{cases}$$



$$\begin{cases} bc(b+c)=1 \\ \cancel{b^2-2bc+c^2+bc=3} \\ b^2+bc+c^2=3 \end{cases}$$

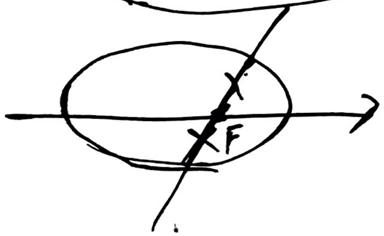
$\boxed{st=1}$

$\boxed{s^2-t=3}$

$$\begin{cases} \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \\ x=ty+m \end{cases}$$

$\frac{ep}{1-e\cos}$

$\frac{ep}{1+e\cos}$



$b^2+2bc+c^2+b^2+b+c-c$

$=2b^2+2bc+c^2+6$

$=(b-c)^2+6$

$b+c=s$

$bc=t$

$x=b-c^2$, $y=c-b^2$

$b-c^2+c-b^2$

$\sqrt{s^2-4t}$

$c-b$

$x+y=$ ——

$x-y=b-c-c^2+b^2$

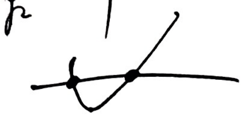
$=(b-c)(b+c+1)$

$\left|\frac{y_1}{y_2}\right|+\left|\frac{y_2}{y_1}\right|=\left|\frac{y_1^2+y_2^2}{y_1y_2}\right|=\left|\frac{(y_1+y_2)^2-2y_1y_2}{y_1y_2}\right|$

u

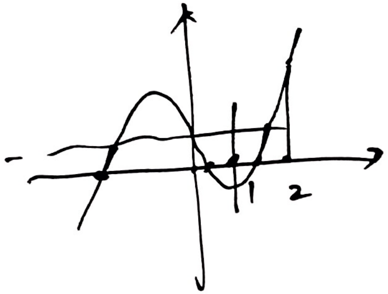
y_1+y_2 y_1y_2

$u+\frac{1}{u}$



$x^3-3x+1=0$.

$b-c^2+6$.



$(-2,2)$

$f(-2)<0$. $f(-1)>0$.

$a\in(-2,-1)$

$-2<a<-1$, $0<b<1<c<2$.

$x=2\cos\theta$.

$x=2\sin\theta$.

$8\cos^3\theta-6\cos\theta+1=0$

$4\cos^3\theta-3\cos\theta=-\frac{1}{2}$.

$\boxed{\cos3\theta=-\frac{1}{2}}$.

$\sin3\theta=3\sin\theta-4\sin^3\theta$.

$3\theta=2k\pi+\frac{2}{3}\pi$.

或 $3\theta=2k\pi-\frac{2}{3}\pi$.

$\theta=\frac{2k}{3}\pi+\frac{2}{9}\pi$.

或 $\theta=\frac{2}{3}k\pi-\frac{2}{9}\pi$.

$a=2\cos\frac{8}{9}\pi$

$b=2\cos\frac{4}{9}\pi$.

$\theta=\frac{2}{9}\pi$ $c=2\cos\frac{2}{9}\pi$.

$\frac{4}{9}\pi$

$\frac{8}{9}\pi$.

~~2cos~~

$2\cos2\alpha-4\cos^2\alpha+6$.

$=2(2\cos^2\alpha-1-2\cos^2\alpha)+6$

$=4$.

5、$n\in N^*$，$c_1, c_2, \cdots, c_{n-1}\in R$，且满足 $\sum_{i=1}^{n-1}|c_i-1|<1$，证明：

方程 $2x^n-c_{n-1}x^{n-1}+c_{n-2}x^{n-2}-\cdots-c_1x^1+2=0$ 无实根.

n 偶.

$2x^2-bx-x+2=0$

$2x^2-x+2=bx$

$|b|<1$

$|bx|<x$

$2x^2-x+2>x$

$2x^2+2(1-x)+x$

$>x$

~~$9x^2-1\cdot x+10=0$~~

$\boxed{2x^2-cx+2=0} \Leftrightarrow 2\left(\frac{1}{x}\right)^2-c\cdot\frac{1}{x}+2=0$

$\forall x$ 是根，$\frac{1}{x}$ ✓.

10

$|c-1|<1$

$c\in(0,2)$

$b=c-1$

$c=b+1$

不妨设 $x\in(0,1)$

~~不妨~~ 否则用 $\frac{1}{x}$ 替换 x，原方程仍成立

x，若 $x\in(0,1)$ $\Rightarrow$ 矛盾！

若 $x\in(1,+\infty)$ $\Rightarrow$ $\frac{1}{x}$ 也是根，$y=\frac{1}{x}\in(0,1)$ 也是根.

$2x^2-cx+2=0$

设 $b=c-1$, $c=b+1$

则 $|b|<1$

方程 $\Leftrightarrow 2x^2-(b+1)x+2=0$

$\Leftrightarrow 2x^2-x+2=bx$ ①

若 $x\in(0,1]$,

则 $bx\le|bx|<x\le x+2(1-x)+2x^2$

$=2x^2-x+2$.

而 $2x^2-x+2$ 矛盾!

不妨设 $x\in(0,1]$,

不一

若 $x\in[1,+\infty)$

方程 $\Leftrightarrow 2x^2-(b+1)x+2=0$

$\Leftrightarrow 2-(b+1)\cdot\frac{1}{x}+2\cdot\frac{1}{x^2}=0$.

令 $y=\frac{1}{x}$, 则 $y\in(0,1]$.

方程 $\Leftrightarrow 2y^2-(b+1)y+2=0$.

$by\le|by|<y\le y+2(1-y)+2y^2$

$=2y^2-y+2$ 矛盾!

不妨设 $x\in(0,1)$

设 $b_i=c_i-1$，$c_i=b_i+1$

$\sum_{i=1}^{n-1}|b_i|<1$.

$2x^n-(b_{n-1}+1)x^{n-1}+(b_{n-2}+1)x^{n-2}-\cdots-(b_1+1)x+2=0.$

原方程 $\Leftrightarrow 2x^n-x^{n-1}+x^{n-2}-\cdots-x+2=\underline{b_{n-1}x^{n-1}-b_{n-2}x^{n-2}+\cdots+b_1x}$.

$\text{RHS}\le\sum_{i=1}^{n-1}|b_ix^i|\le\sum_{i=1}^{n-1}|b_i|x<x$.

下

$-x^7+2x^8>x.$

$\underline{2-x}+\underline{x^2-x^3}+\underline{x^4-x^5}+\underline{\cancel{x^6}-\cancel{x^7}}+2x^8\ge x$ 矛盾.

$2(1-x)+x+x^2(1-x)+x^4(1-x)+x^6(1-x)+2x^8$

$\underline{2(1-x)+x}+\underline{x^2(1-x)}+\underline{x^4(1-x)}+\cdots$

$+x^{n-2}(1-x)+x^{n-4}(1-x)+\cdots+x^2(1-x)+2(1-x)+x>x$.

$\text{LHS}=2x^n+x^{n-2}(1-x)+x^{n-4}(1-x)+\cdots$

6、设 $n\ge 2$，$n\in\mathbb{N}^*$，$a_1,a_2,\cdots,a_n\in\mathbb{R}^+$，且满足

$(a_1+a_2+\cdots+a_n)\left(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n}\right)\le\left(n+\frac{1}{2}\right)^2$.

证明：$\max\{a_1,a_2,\cdots,a_n\}\le 4\min\{a_1,a_2,\cdots,a_n\}$

不妨 $a_1\le a_2\le\cdots\le a_n$.

证 $\frac{a_n}{a_1}\le 4$，设 $a_n=ta_1$，证 $t\le 4$

记 $\begin{cases} a_2+\cdots+a_{n-1}=X. \\ \frac{1}{a_2}+\cdots+\frac{1}{a_{n-1}}=Y \end{cases}$

$XY\ge(n-2)^2$

$(a_1+a_n+X)\left(\frac{1}{a_1}+\frac{1}{a_n}+Y\right)\le\left(n+\frac{1}{2}\right)^2$.

$n^2+n+\frac{1}{4}\ge 2+t+\frac{1}{t}+X\cdot\frac{a_1+a_n}{a_1a_n}+Y(a_1+a_n)+XY$

$\ge 2+t+\frac{1}{t}+\cancel{X\cdot\frac{a_1+a_n}{a_1a_n}}\ \cancel{+Y(a_1+a_n)}+(n-2)^2$.

$\ge 2+t+\frac{1}{t}+\frac{t+1}{ta_1}X+Y(t+1)a_1$.

$\ge$

$(2s+4n-3)(2s-5)\le 0$.

$\ge 2+t+\frac{1}{t}$

$+(n-2)^2+2\sqrt{\frac{(t+1)^2}{t}XY}$

$\ge 2+t+\frac{1}{t}+(n-2)^2$

$+2(n-2)\frac{t+1}{\sqrt{t}}$.

$s=\sqrt{t}+\frac{1}{\sqrt{t}}$.

$s^2=t+\frac{1}{t}+2$.

$s^2+\cancel{n^2}-5n+\frac{15}{4}$

$+2(n-2)s\le\cancel{n^2}$

$s^2+2(n-2)s-5n+\frac{15}{4}\le 0$

$(2s)^2+(4n-8)(2s)-20n+15\le 0$.

$4n-3$.

$1\qquad -5$

1

$2S \le 5.$

$\sqrt{t}+\frac{1}{\sqrt{t}} \le \frac{5}{2}.$

$\sqrt{t} \in [\frac{1}{2}, 2]$

$t \in [1, 4]$

$\frac{a_n}{a_1} \le 4$

$(n+\frac{1}{2})^2 \ge (a_1+a_2+\cdots+a_n)(\frac{1}{a_1}+\frac{1}{a_2}+\cdots+\frac{1}{a_n})$

$= \left(\underline{(a_1+a_n)}+a_2+\cdots+a_{n-1}\right)\left(\left(\frac{1}{a_1}+\frac{1}{a_n}\right)+\frac{1}{a_2}+\cdots+\frac{1}{a_{n-1}}\right)$

$\ge \left(\sqrt{(a_1+a_n)(\frac{1}{a_1}+\frac{1}{a_n})}+n-2\right)^2$

$= \underline{\left(\sqrt{t+\frac{1}{t}+2}+n-2\right)^2}.$

$\sqrt{t+\frac{1}{t}+2} \cancel{+n-2} \le n+\frac{1}{2}+2.$

$\sqrt{t+\frac{1}{t}+2} \le \frac{25}{4}.$

$t+\frac{1}{t}+2 \le \frac{25}{4}$

$t+\frac{1}{t} \le \frac{17}{4}$

$t \le 4.$