$\frac{1}{2}\left(S_{\triangle H_2H_6H_3} - S_{\triangle H_2H_6H_4}\right)$

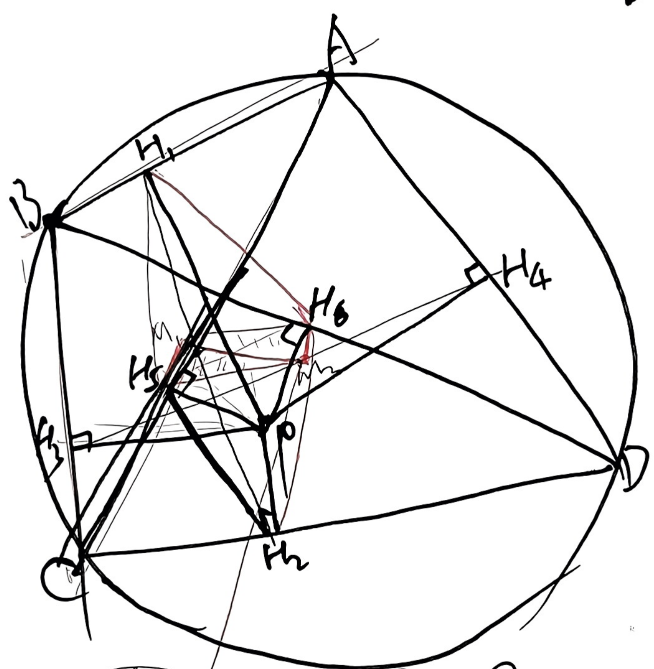
$= \frac{1}{2}\left(S_{\triangle H_1H_5H_4} - S_{\triangle H_1H_5H_3}\right) + \frac{1}{2}\left(S_{\triangle H_5H_2H_4} - S_{\triangle H_5H_2H_3}\right)$

$\Longleftrightarrow S_{\triangle H_6H_1H_3} - S_{\triangle H_6H_1H_4} + S_{\triangle H_2H_6H_3} + S_{\triangle H_2H_6H_4}$

$= S_{\triangle H_1H_5H_4} - S_{\triangle H_1H_5H_3} - S_{\triangle H_5H_2H_4} + S_{\triangle H_5H_2H_3}$

$\Longleftrightarrow S_{\triangle H_6H_1H_3} + S_{\triangle H_2H_6H_3} + S_{\triangle H_1H_5H_3} + S_{\triangle H_5H_2H_4}$

$= S_{\triangle H_1H_5H_4} + S_{\triangle H_5H_2H_3} + S_{\triangle H_6H_1H_4} + S_{\triangle H_2H_6H_4}$



$S_{\triangle H_2H_6H_3} = \frac{R^2-OP^2}{4R^2} S_{\triangle BCD}$

$S_{\triangle H_1H_5H_3} = \frac{R^2-OP^2}{4R^2} S_{\triangle ABC}$

$S_{\triangle ABD}$

$S_{\triangle H_2H_6H_4} =$

$S_{\triangle H_5H_2H_4} = \frac{R^2-OP^2}{4R^2} S_{\triangle ACD}$

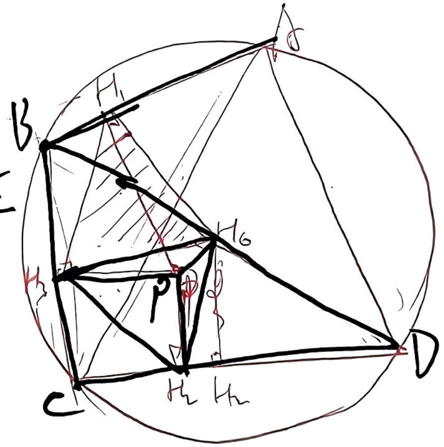
$S_{\triangle H_1H_6H_4} = \frac{R^2-OP^2}{4R^2} S_{\triangle ABD}$

消M_3 $S_{\triangle M_1M_2H_6} = S_{\triangle M_1M_2H_5}$

消M_1 $\frac{1}{2}\left(S_{\triangle M_2H_6H_1} + S_{\triangle M_2H_6H_2}\right)$

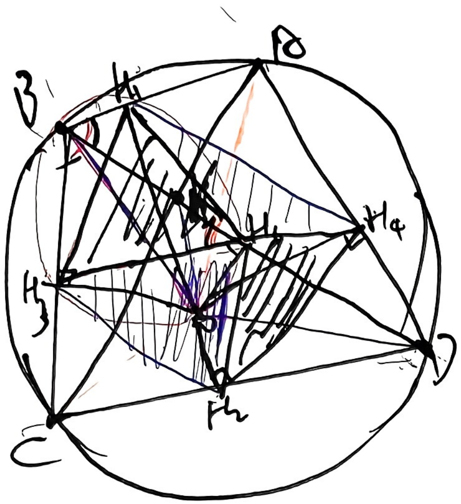
$= \frac{1}{2}\left(S_{\triangle M_2H_5H_1} - S_{\triangle M_2H_5H_2}\right)$

消M_2 $\frac{1}{2}\left(S_{\triangle H_6H_1H_3} - S_{\triangle H_6H_1H_4}\right)$



$\Longleftrightarrow S_{\triangle H_6H_1H_3} + S_{\triangle H_2H_6H_4} = S_{\triangle H_1H_5H_4} + S_{\triangle H_5H_2H_3}$

16



$$\frac{AD\cdot CD\cdot AC}{4R} = S_{\triangle ACD}$$

$$S_{\triangle H_1H_3H_6} = \frac{1}{2} H_1H_6\cdot H_3H_6\cdot \sin\angle H_1H_6H_3$$

$$= \frac{1}{2} BP\sin\angle ABD\cdot \sin B$$

$$BP\sin\angle CBD$$

$$= \frac{1}{2} BP^2 \sin\angle ABD \sin\angle CBD\cdot \sin B$$

$$= \frac{1}{2} BP^2\cdot \frac{AD}{2R}\cdot\frac{CD}{2R}\cdot\frac{AC}{2R}$$

$$= \frac{1}{2} BP^2\cdot \frac{S_{\triangle ACD}}{2R^2}$$

同理 $S_{\triangle H_2H_6H_4} = \frac{1}{2} DP^2\cdot \frac{S_{\triangle ABC}}{2R^2}$

$$S_{\triangle H_1H_5H_4} = \frac{1}{2} AP^2\cdot \frac{S_{\triangle BCD}}{2R^2}$$

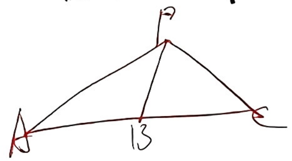
$$S_{\triangle H_5H_2H_3} = \frac{1}{2} CP^2\cdot \frac{S_{\triangle ABD}}{2R^2}$$

只需证

$$\frac{BP^2\cdot S_{\triangle ACD}}{S_{ABCD}} + \frac{DP^2\, S_{\triangle ABC}}{S_{ABCD}} = \frac{AP^2\cdot S_{\triangle BCD}}{S_{ABCD}} + \frac{CP^2\cdot S_{\triangle ABD}}{S_{ABCD}}$$

$$\iff BP^2\cdot\frac{DK}{BD} + DP^2\cdot\frac{BK}{BD} = AP^2\cdot\frac{CK}{AC} + CP^2\cdot\frac{AK}{AC}$$

$$\iff \frac{BP^2\cdot DK + DP^2\cdot BK}{BD} = \frac{AP^2\cdot CK + CP^2\cdot AK}{AC}$$

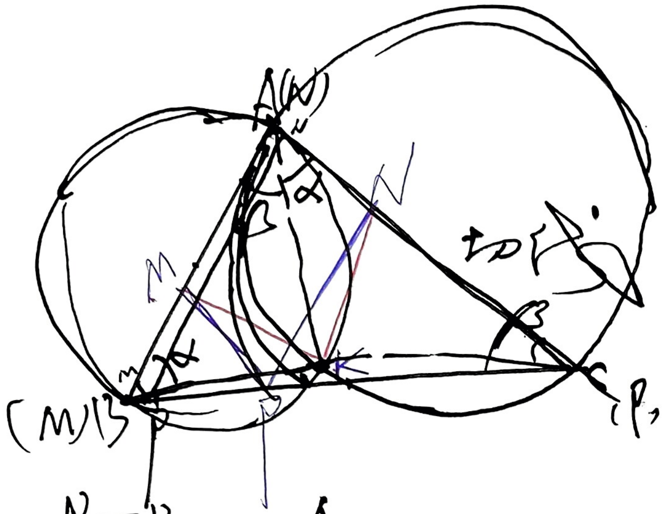


$$PA^2\cdot BC + PC^2\cdot AB = PB^2\cdot AC + AB\cdot BC\cdot AC$$

斯氏

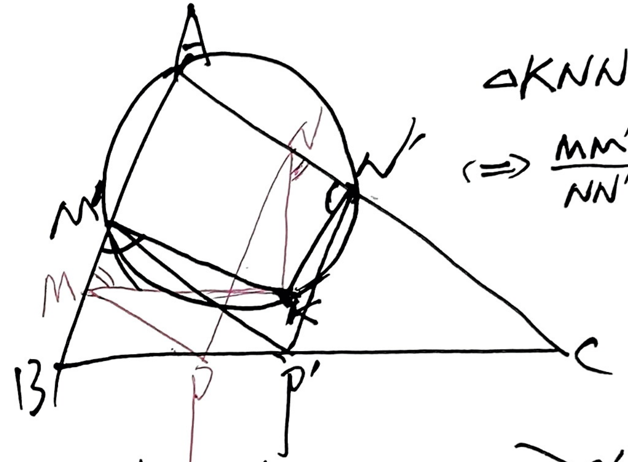
$$\frac{PK^2\cdot BD + BK\cdot DK\cdot BD}{BD} = \frac{PK^2\cdot AC + AK\cdot CK\cdot AC}{AC}$$

$$\iff PK^2 + BK\cdot DK = PK^2 + AK\cdot CK$$



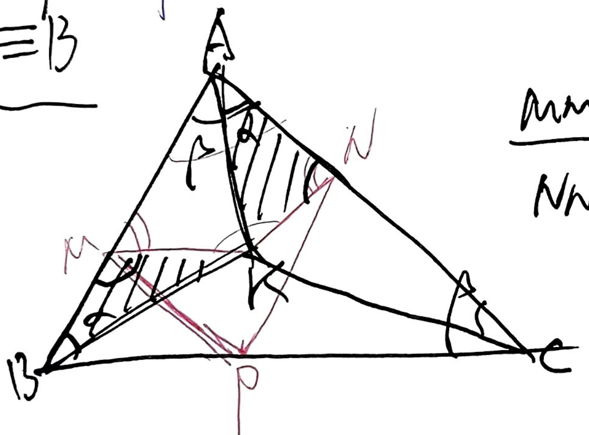
$P \equiv B$

二



$\triangle KNN' \sim \triangle KMM'$

$\Longleftrightarrow \dfrac{MM'}{NN'} = \dfrac{M'K}{N'K}$



$$\frac{AM'}{NN'} = \frac{MM'}{PP'} \cdot \frac{PP'}{NN'} = \frac{BM'}{\cancel{BP'}} \cdot \frac{\cancel{CP'}}{CN'} = \frac{BM'}{CN'}$$

$$= \frac{AM'}{AN'}$$

只需证 $\dfrac{AM'}{AN'} = \dfrac{M'K}{N'K}$

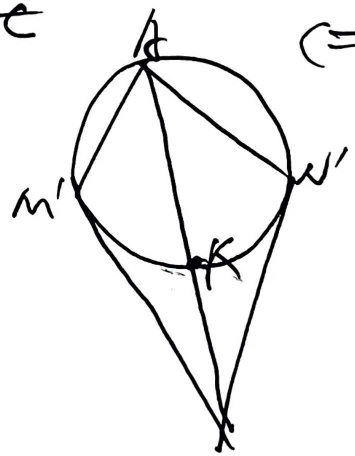
$\Longleftrightarrow AM' \cdot N'K = M'K \cdot AN'$

可用调和四边形

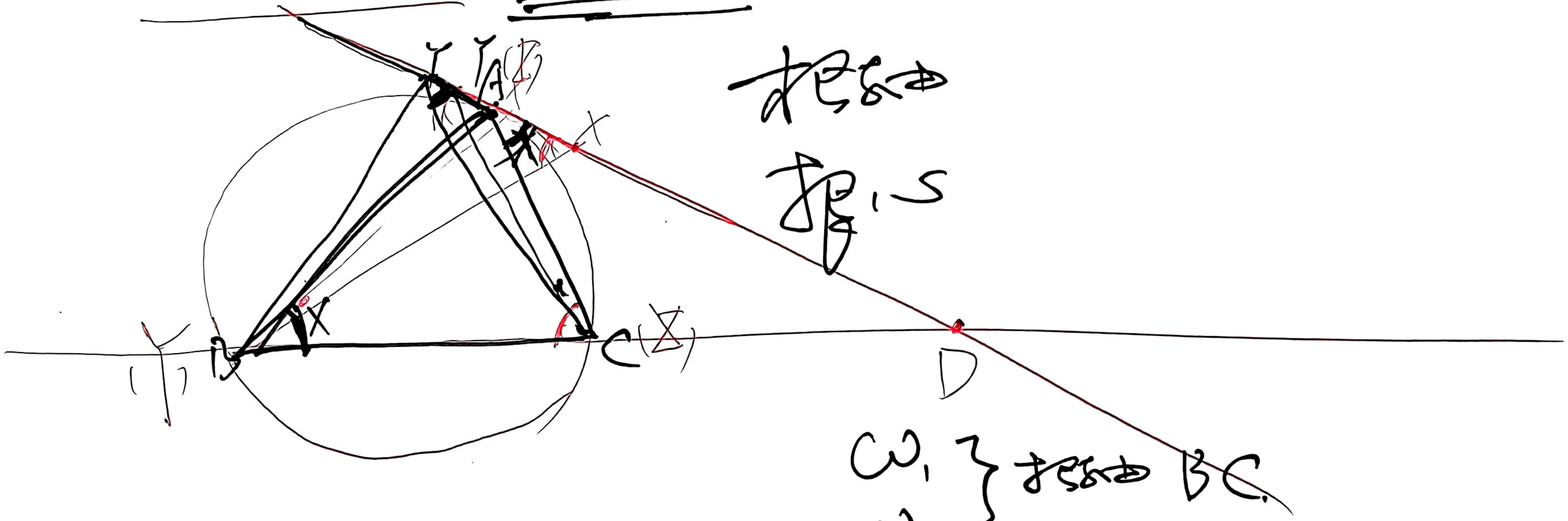
$\triangle ABK \sim \triangle CAK$

$\dfrac{AM}{BM} = \dfrac{CP}{BP} = \dfrac{CN}{AN}$

M、N 对于定点



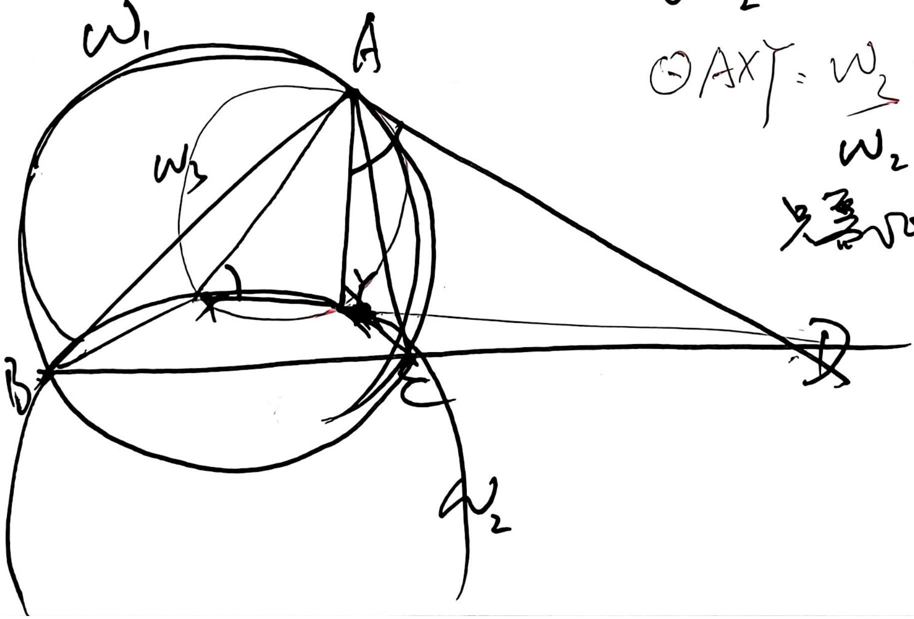
$\angle AXB - \angle ACB = \angle AYC - \angle ABC = 0$



根轴

根1S

$\left.\begin{matrix}\omega_1\\ \omega_2\end{matrix}\right\}$ 根轴 BC.

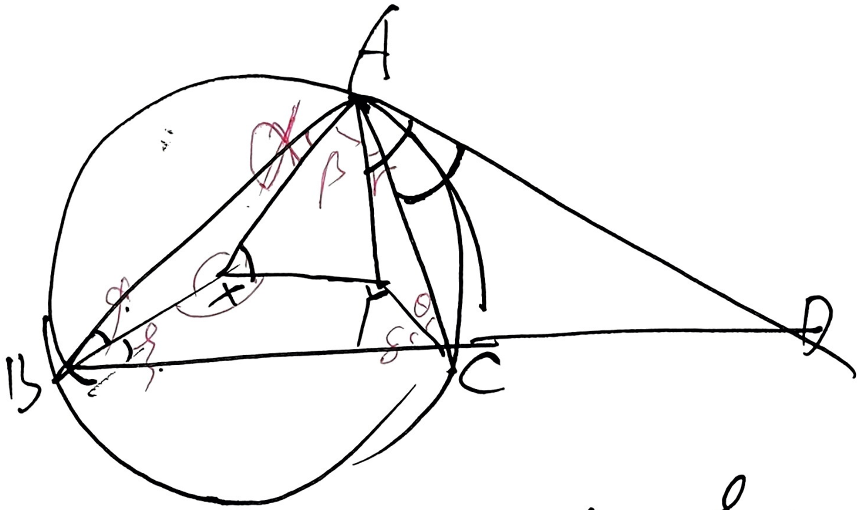


$\left.\begin{matrix}\odot AXY = \omega_3\\ \omega_2\end{matrix}\right\}$ 根轴: XY

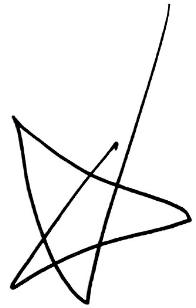
只需证 ω_1、ω_3 根轴为 AD

AD 与 ω_1 相切

只需证 AD 与 ω_3 相切



$$\left.\begin{aligned}\angle AXB-\angle ACB&=\beta+\gamma+\xi\\ \angle AYC-\angle ABX&=\beta+\alpha+\delta\end{aligned}\right\}\Rightarrow \gamma+\xi=\alpha+\delta$$



要证 $\angle DAY=\angle AXY=360-\angle AXB-\angle BXY$

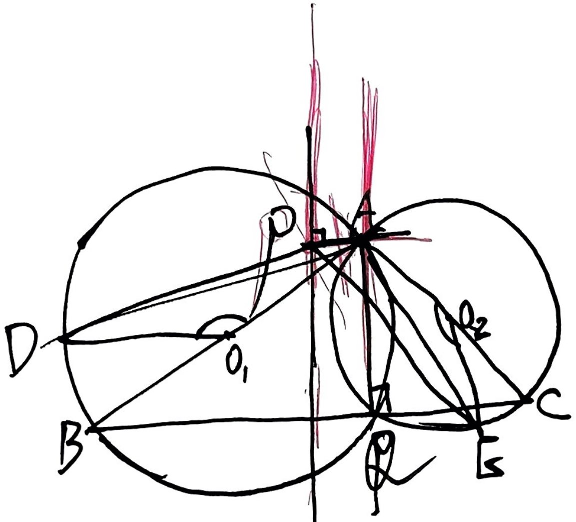
$\angle DAY=\gamma+\varphi+\xi$

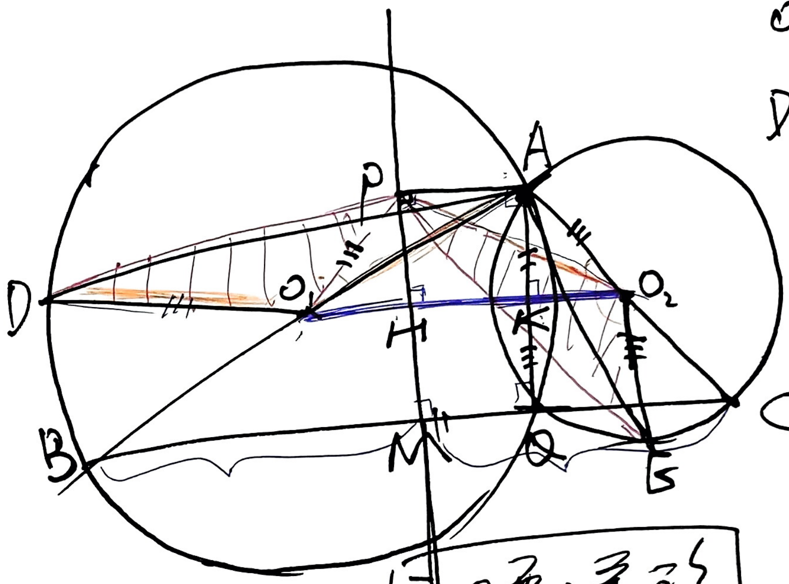
$$=360-(180-\alpha-\varphi)-(180-\delta)$$

$$=\alpha+\varphi+\delta$$

即要证 $\gamma+\varphi+\xi=\alpha+\varphi+\delta$

$$\Longleftrightarrow \gamma+\xi=\alpha+\delta$$





$O_1P = O_2A = O_2E$

$DO_1 = O_1A = PO_2$

$\angle DO_1P = \angle AO_1D - \angle AO_1P$

$\angle EO_2P = \angle AO_2E - \angle AO_2P$

$\therefore \triangle DO_1P \cong \triangle PO_2E$

$\therefore \underline{PD = PE}$

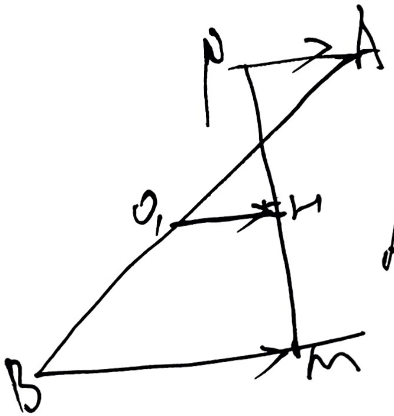
构造中位线 A

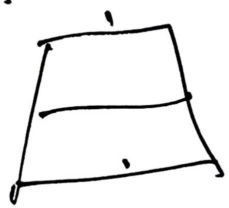
只需证 $PD = PE$

证 APO_1O_2 为 $\boxed{\text{等腰梯形}}$

证 $\triangle PHO_1 \cong \triangle AKO_2$

$PH = AK \quad \angle PHO_1 = \angle AKO_2 = 90^\circ$





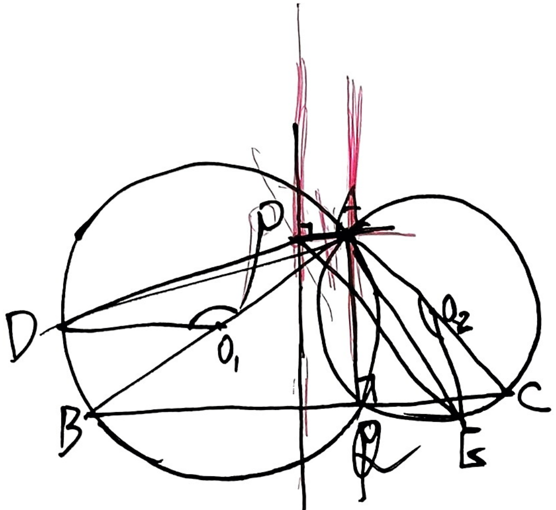
下证 $O_1H = O_2K$

$O_2K = \frac{1}{2}CQ$

$O_1H = \frac{1}{2}(BM - AP)$,

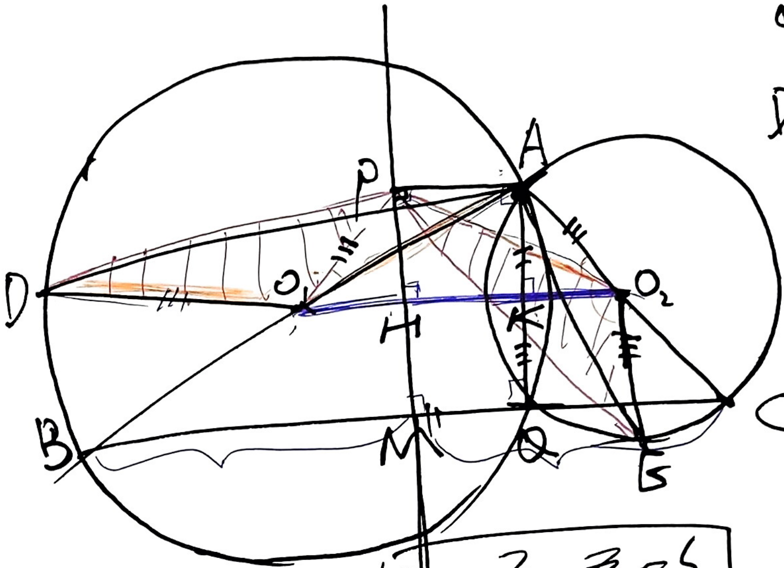
只需证 $CQ = BM - AP \Leftrightarrow CQ + AP = BM$

$\Leftrightarrow CQ + MQ = BM \Leftrightarrow CM = BM$ ✓



相错情况 A

只需证 $PD=PE$



$O_1P=O_2A=O_2E$

$DO_1=O_1A=PO_2$

$\angle DO_1P=\angle AO_1D-\angle AO_1P$

$\angle EO_2P=\angle AO_2E-\angle AO_2P$

$\therefore \triangle DO_1P\cong\triangle PO_2E$

$\therefore PD=PE$

证 APO_1O_2 为 $\boxed{\text{平行四边形}}$

证 $\triangle PHO_1\cong\triangle AKO_2$

$PH=AK \quad \angle PHO_1=\angle AKO_2=90^\circ$

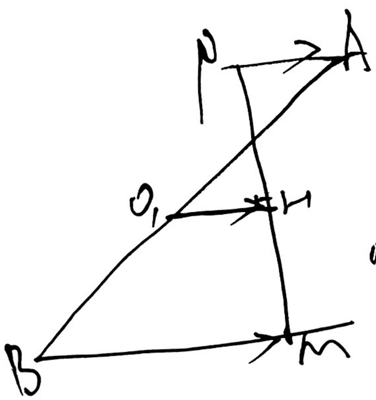
下证 $O_1H=O_2K$

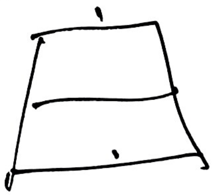
$O_2K=\frac{1}{2}CQ$

$O_1H=\frac{1}{2}(BM-AP)$，

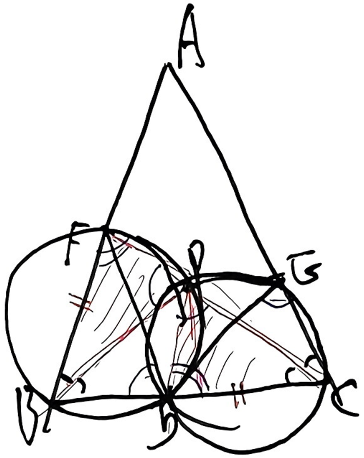
只需证 $CQ=BM-AP \Leftrightarrow CQ+AP=BM$

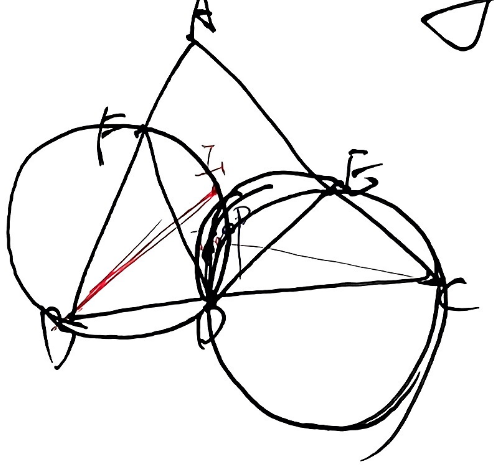
$\Leftrightarrow CQ+MQ=BM \Leftrightarrow CM=BM$ ✓



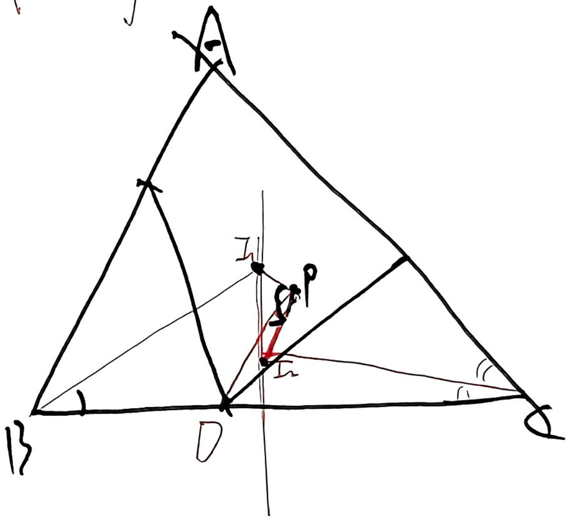


有向角



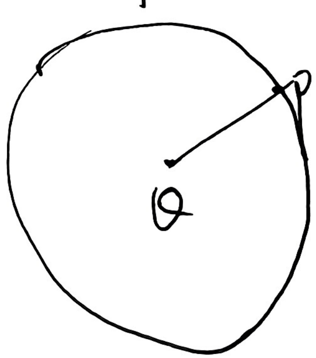


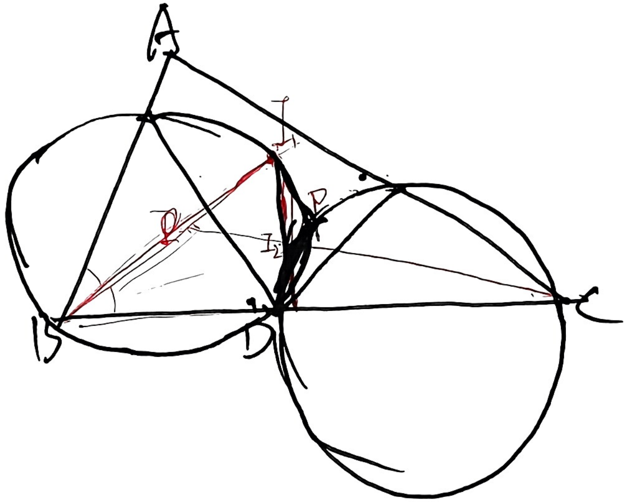
∠I_1PI_2为定值

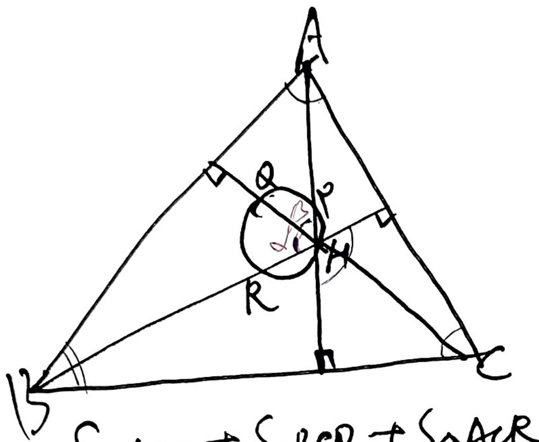


$\triangle BFP \cong \triangle CDP$

$\Rightarrow FP = DP$







$S_{\triangle ABQ}+S_{\triangle BCP}+S_{\triangle ACR}$

$=S_{\triangle ABC}$，H为垂心.

若B. H.P.Q.R共圆.

$S_{\triangle ABQ}+S_{\triangle BCP}+S_{\triangle ACR}$

$=S_{\triangle ABH}+S_{\triangle BCH}+S_{\triangle ACH}$

$(S_{\triangle BCP}-S_{\triangle BCH})+(S_{\triangle ACR}-S_{\triangle ACH})=(S_{\triangle ABH}-S_{\triangle ABQ})$

$BC\cdot PH+AC\cdot HR=AB\cdot HQ$

$\sin A\cos A=\sin C\cos C+\sin B\cos B-2\sin A\cos B\cos C$

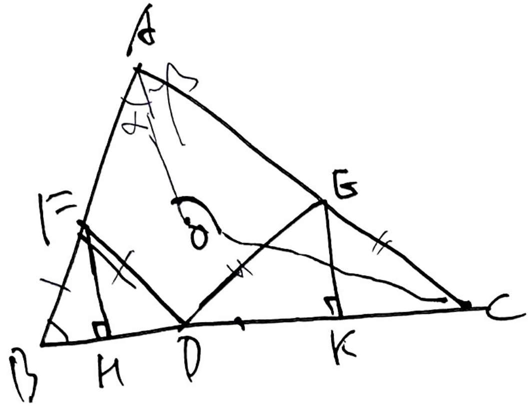
$\Longleftrightarrow \sin A\,(\cos A+2\cos B\cos C)=\frac{1}{2}\sin 2C+\frac{1}{2}\sin 2B$

$\Longleftrightarrow \sin A\,[-\cos(B+C)+2\cos B\cos C]=\frac{1}{2}(\sin 2C+\sin 2B)$

$\Longleftrightarrow \sin A\cos(B-C)=\frac{1}{2}\cdot 2\sin\frac{2C+2B}{2}\cos\frac{2C-2B}{2}$

需证 $PH\sin\alpha+HR\sin\beta=HQ\sin(\alpha+\beta)$

$PH\sin A+HR\sin B=HQ\sin C$



$$OA\sin A = AF\sin\beta + AE\sin\alpha$$

$$AF\sin(90^\circ - B) + AE\sin(90^\circ - C)$$

$$\iff OA\sin A = AF\cos B + AE\cos C$$

$$= (AB - BF)\cos B + (AC - CE)\cos C$$

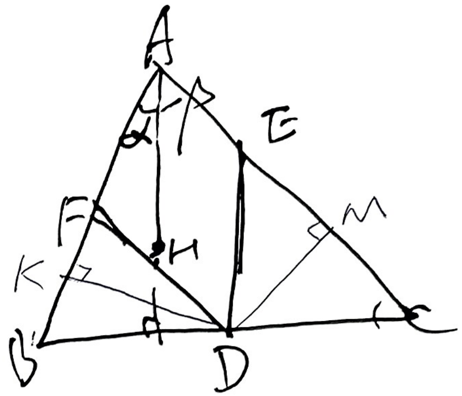
$$= AB\cos B + AC\cos C - (BF\cos B + CE\cos C)$$

$$\iff OA\sin A = \underline{AB\cos B + AC\cos C} - \frac{1}{2}(BD + CD)$$

$$BC$$

$$R\sin A = \frac{1}{2}\cdot \underline{2R\sin A}$$

$$= \frac{1}{2}\cdot BC$$



$$AH\sin A = AF\sin\beta + AE\sin\alpha$$

$$\iff \underline{2R\cos A\sin A} = (AB - 2BK)\cos C + (AC - 2CM)\cos B$$

$$\iff BC\cos A = AB\cos C + AC\cos B \underline{- 2BK\cos C - 2CM\cos B}$$

$$\parallel$$

$$-2BD\cos B\cos C - 2CD\cos C\cos B$$

$$= -2\cos B\cos C\cdot BC$$

$$\iff BC\cos A = AB\cos C + AC\cos B - 2BC\cos B\cos C$$