3. 求证 对任意 $r\in\mathbb{Q}_+$，则 $\frac{n^{r+1}}{r+1} \overset{①}{<} S_r(n) \overset{②}{<} \frac{(n+1)^{r+1}}{r+1}$.

伯努利不等式：$x>-1$，①若 $\alpha\le 0$ 或 $\alpha\ge 1$，则 $(1+x)^\alpha \ge 1+\alpha x$

②若 $0\le\alpha\le 1$，则 $(1+x)^\alpha \le 1+\alpha x$

先证①. $n=1$ 时. ①左 $=\frac{1}{r+1} < 1 =$ ①右

假设对 $n\in\mathbb{Z}_+$. ①成立. 对于 $n+1$. 由归纳假设可得

$$S_r(n+1)=S_r(n)+(n+1)^r > \frac{n^{r+1}}{r+1}+(n+1)^r \overset{只需证}{\ge} \frac{(n+1)^{r+1}}{r+1}$$

$$\Updownarrow$$

$$n^{r+1}+(r+1)(n+1)^r \ge (n+1)^{r+1}$$

$$\Updownarrow$$

$$\left(1-\frac{1}{n+1}\right)^{r+1} \ge 1+(r+1)\left(-\frac{1}{n+1}\right)$$

再证② $n=1$ 时 只需证 $1<\frac{(1+1)^{r+1}}{r+1} \Leftrightarrow (1+1)^{r+1} > r+1$

由伯努利不等式 左 $\ge 1+(r+1)\cdot 1 > r+1$

假设对 $n\in\mathbb{Z}_+$. ②成立. 对于 $n+1$ 由归纳假设可得

$$S_r(n+1)=S_r(n)+(n+1)^r<\frac{(n+1)^{r+1}}{r+1}+(n+1)^r\le\frac{(n+2)^{r+1}}{r+1}$$

$$\Updownarrow$$

$$(n+1)^{r+1}+(r+1)(n+1)^r\le(n+2)^{r+1}$$

$$\Updownarrow$$

$$1+(r+1)\cdot\frac{1}{n+1}\le\left(1+\frac{1}{n+1}\right)^{r+1}$$

由引理可得

$$\frac{n^{a+1}}{a+1}<S_a(n)=(S_b(n))^c<\left(\frac{(n+1)^{b+1}}{b+1}\right)^c$$

$$\Rightarrow\frac{(b+1)^c}{a+1}<\frac{(n+1)^{(b+1)c}}{n^{a+1}} \quad ③$$

$$\left(\frac{n^{b+1}}{b+1}\right)^c<(S_b(n))^c=S_a(n)<\frac{(n+1)^{a+1}}{a+1}$$

$$\Rightarrow\frac{n^{(b+1)c}}{(n+1)^{a+1}}<\frac{(b+1)^c}{a+1} \quad ④$$

由③、④可知 $\frac{n^{(b+1)c}}{(n+1)^{a+1}}<\frac{(b+1)^c}{a+1}<\frac{(n+1)^{(b+1)c}}{n^{a+1}}$ ⑤

3/

若 $(b+1)c>a+1$ ~~[illegible]~~ ④左 $\to +\infty$ $(n\to+\infty)$ 由④可得矛盾

若 $(b+1)c<a+1$ ③左 $\to 0$ $(n\to+\infty)$，由③可得矛盾

$\therefore (b+1)c=a+1$ ⑥ $\therefore$ ⑤左 $\to 1$ $(n\to+\infty)$，⑤左 $\to 1$ $(n\to+\infty)$

$\therefore \frac{(b+1)^c}{a+1}=1$ $\therefore (b+1)^c=a+1$ ⑦

若 $c=1$，由⑥可得 $a=b\in\mathbb{Q}_+$ ✓

若 $c\geq 2$，由⑥⑦可得 $(b+1)^c=(b+1)\cdot c \Rightarrow (b+1)^{c-1}=c\in\mathbb{Z}_+$ ⑧

$\Rightarrow b+1\in\mathbb{Z}_+ \Rightarrow b\in\mathbb{Z}_+$ 由⑧可得 $c=(b+1)^{c-1}\geq 2^{c-1}$

$\Rightarrow c=2$ $\therefore b=1$ $a=3$ ✓

36. $3\sum a^2\geq\sum a\sum\sqrt{bc}+\sum(a-b)^2 \Leftrightarrow (\sum a)^2\geq\sum a\sum\sqrt{bc}$

$2(a^2+b^2+c^2)-2(ab+bc+ca)$ $\Uparrow$

$(x^2+y^2+z^2)\geq xy+yz+zx$ $\Leftrightarrow \sum(x-y)^2\geq 0$

$\sum(\sqrt{b}-\sqrt{c})^2\geq 0 \Leftrightarrow \sum a\geq\sum\sqrt{bc}$，$\sum a\geq 0$ ✓

88

设 $a=x^2\ b=y^2\cdot c=z^2\cdot\ x,y,z\geqslant 0$ 则 $\underline{\sum x^4+xyz\sum x+\sum xy(x^2+y^2)}$

$\sum a\sum\sqrt{bc}+\sum(a-b)^2\geqslant(\sum a)^2$ 化为

$\geqslant 4\sum x^2y^2$ ①

舒尔不等式 若 $x,y,z>0$ 且 $r\in R$，则 $\sum x^r(x-y)(x-z)\geqslant 0$

$\therefore \sum x^2(x-y)(x-z)\geqslant 0 \Leftrightarrow \sum(x^4-(y+z)x^3+yzx^2)\geqslant 0.$

$\Leftrightarrow \sum x^4+xyz\sum x\geqslant\sum(y+z)x^3=\sum xy(x^2+y^2)$

$\therefore$ ① 左 $\geqslant 2\sum xy(x^2+y^2)\geqslant 4\sum x^2y^2$.

前部由均值不等式

37 (1) $n=1$ 时 $b_1\geqslant 2>\frac{2\sqrt{2}}{3}$ ✓ 假设 $n=k\in Z_+$ 时结论成立

$n=k+1$ 时 假设 $2\leqslant b_i\leqslant\sqrt{2(k+1)}$ $(i=1,2\cdots k+1)$

设 $m=[\sqrt{2(k+1)}]$ 则 $2\leqslant b_i\leqslant m$ $(i=1,2\cdots k+1)$

对于每个 $b_i = j \in \{2, 3, \cdots, m\}$，则 $a_i = 1, 2, \cdots, j-1$.

所有可能 $\frac{a_i}{b_i}$ 共有 $1+2+\cdots+m-1 = \frac{(m-1)\cdot m}{2}$ 种

$\therefore k+1 \leq \frac{(m-1)m}{2} < \frac{m^2}{2}$ $\therefore m > \sqrt{2(k+1)} \geq M$ 矛盾

$\therefore$ 存在 $i_0 \in \{1, 2, \cdots, k+1\}$ 使得 $b_{i_0} > \sqrt{2(k+1)}$（由归纳假设）

$\therefore$ 可得 $\sum_{i=1}^{k+1} b_i = \sum_{\substack{i=1 \\ i \neq i_0}}^{k+1} b_i + b_{i_0} \geq \frac{2\sqrt{2}}{3} k^{\frac{3}{2}} + \sqrt{2(k+1)} \overset{\text{要证}}{\geq} \frac{2\sqrt{2}}{3}(k+1)^{\frac{3}{2}}$

$\Uparrow$

$2k\sqrt{k} + 3\sqrt{k+1} \geq 2(k+1)\sqrt{k+1}$ ①

$\Uparrow$

$\sqrt{k+1} \geq 2k(\sqrt{k+1} - \sqrt{k}) = \frac{2k}{\sqrt{k+1}+\sqrt{k}}$

$\Uparrow$

$\sqrt{k+1}(\sqrt{k+1} + \sqrt{k}) \geq 2k$ ✓

(2) $n=1$ 时 $b_1 \geq 2 \geq 2\left(\frac{2}{3}\right)^{\frac{3}{2}}$ ✓ 假设 $n=k$ 时 ✓ 对于 $n=k+1$ 时

假设对于每个 $i \in \{1, 2, \cdots, k+1\}$，均有 $2 \leq b_i \leq \frac{2\sqrt{6}}{3}\sqrt{k+1}$

设 $m=\left[\frac{2\sqrt{6}}{3}\sqrt{k+1}\right]$ 则 $2\le b_i\le m\ (i=1,2\cdots k+1)$

$\left[\frac{m}{2}\right]\ge\frac{m-1}{2}$

$\therefore k+1\le 1+2+\cdots+m-1-\left(1+2+\cdots+\left(\left[\frac{m}{2}\right]-1\right)\right)$

$=\frac{(m-1)m}{2}-\frac{\left(\left[\frac{m}{2}\right]-1\right)\left[\frac{m}{2}\right]}{2}\le\frac{(m-1)m}{2}-\frac{\left(\frac{m-1}{2}-1\right)\left(\frac{m-1}{2}\right)}{2}$

$=\frac{3}{8}(m^2-1)<\frac{3}{8}m^2\quad\therefore m>\sqrt{\frac{8(k+1)}{3}}=\frac{2\sqrt{6}}{3}\sqrt{k+1}\ge m$

矛盾 $\therefore$ 存在 $i_0\in\{1,2\cdots k+1\}$ 使得 $b_{i_0}>\frac{2\sqrt{6}}{3}\sqrt{k+1}$. 由归纳假设

$\sum_{i=1}^{k+1}b_i=\sum_{\substack{i=1\\i\ne i_0}}^{k+1}b_i+b_{i_0}\ge 2\left(\frac{2k}{3}\right)^{\frac{3}{2}}+\frac{2\sqrt{6}}{3}\sqrt{k+1}\ \overset{只需}{\underset{①}{\ge}}\ 2\left(\frac{2(k+1)}{3}\right)^{\frac{3}{2}}$

$\cancel{2}\cdot\frac{2k}{3}\cdot\sqrt{\frac{2k}{3}}+\cancel{2}\sqrt{\frac{2}{3}}\sqrt{k+1}\ge\cancel{2}\cdot\frac{2(k+1)}{3}\sqrt{\frac{2(k+1)}{3}}$

$2k\sqrt{k}+3\sqrt{k+1}\ge 2(k+1)\sqrt{k+1}$ ⑧ ①

38. $\sum\frac{33a^2-a}{33a^2+1}\ge 0 \Leftrightarrow \sum\left(\frac{33a^2-a}{33a^2+1}+t\right)\ge 3t$

待定 $t>0$

$$\Leftarrow \sum \frac{33(t+1)a^2-a+t}{33a^2+1} \geq 3t \quad ①$$

令 $2\sqrt{33(t+1)}\cdot\sqrt{t}=1 \Rightarrow 132t^2+132t-1=0$

得有唯一正根 $t=-\frac{1}{2}+\sqrt{\frac{17}{66}} \in (0,1)$

则 ① 化为 $\sum \frac{(\sqrt{33(t+1)}\,a-\sqrt{t})^2}{33a^2+1} \geq 3t \quad ②$

先 $\because a^2=(-b-c)^2=(b+c)^2 \underset{柯西不等式}{\leq} 2(b^2+c^2)$

$\therefore 3a^2 \leq 2(a^2+b^2+c^2) \Rightarrow 33a^2+1 \leq 22(a^2+b^2+c^2)+1$

$\therefore$ ② 左 $\geq \sum \frac{(\sqrt{33(t+1)}\,a-\sqrt{t})^2}{22(a^2+b^2+c^2)+1} \overset{只需}{\geq} 3t$

$\Updownarrow$

$$\frac{\sum(\sqrt{33(t+1)}\,a-\sqrt{t})^2}{22(a^2+b^2+c^2)+1} \geq 3t$$

$$\Leftrightarrow \sum\left(33(t+1)a^2+t-2\sqrt{33(t+1)}\sqrt{t}\cdot a\right)\geq 3t\cdot\left(\cancel{22}(a^2+b^2+c^2)+1\right)$$

$$\Leftrightarrow \sum 33(t+1)a^2-\sum a\geq 3t\cdot\cancel{22}(a^2+b^2+c^2).$$

$$\Leftrightarrow \cancel{33}(t+1)\sum a^2\geq 3t\cdot\cancel{22}\sum a^2$$

$$\Uparrow$$

$$t+1\geq 2t$$

$$\Updownarrow$$

$$t\leq 1$$

$\sum a^2\geq 0$ ✓

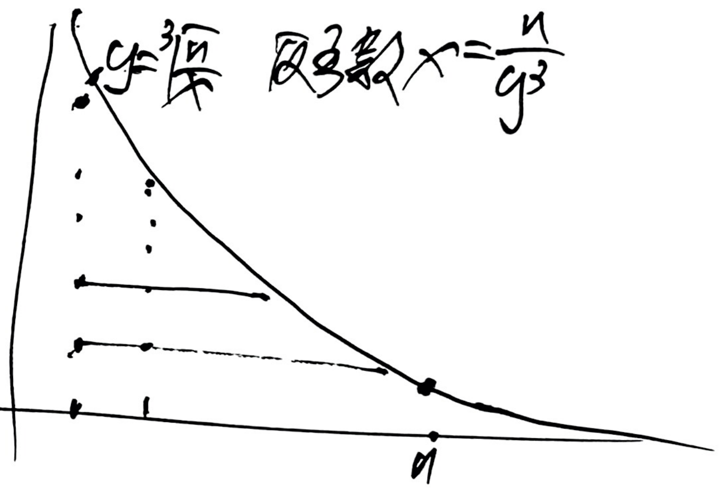
39.

$x=[x]+\{x\}$ $\quad 0\leq\{x\}<1$

设 $y=\sqrt[3]{\frac{n}{x}}$ $(x>0)$

设 $D=\{(x,y)\mid x,y\in Z_+,\ y\leq\sqrt[3]{\frac{n}{x}}\}$

则 $D=\{(x,y)\mid x,y\in Z_+,\ x\leq\frac{n}{y^3}\}$.



$$\therefore \sum_{i=1}^{n}\left[\sqrt[3]{\frac{n}{i}}\right] = \sum_{j=1}^{[\sqrt[3]{n}]}\left[\frac{n}{j^3}\right] \le \sum_{j=1}^{[\sqrt[3]{n}]}\frac{n}{j^3} \overset{只需}{<} \frac{5n}{4}$$

(D)

$$\Updownarrow$$

$$\sum_{j=1}^{[\sqrt[3]{n}]}\frac{1}{j^3} < \frac{5}{4}$$

$\therefore$ 只需证对于任意 $m\in\mathbb{Z}_+$ 均有 $\sum_{j=1}^{m}\frac{1}{j^3} < \frac{5}{4}$

法一：只需证 $\sum_{j=1}^{m}\frac{1}{j^3} \le \frac{5}{4} - \frac{1}{4m}$ ①

$m=1$，①左 $=1=\frac{5}{4}-\frac{1}{4}=$ ①右

假设 $m=k$ 时 ① 成立，则 $m=k+1$ 时

$$\sum_{j=1}^{m+1}\frac{1}{j^3} \le \frac{5}{4} - \frac{1}{4m} + \frac{1}{(m+1)^3} \overset{只需}{\le} \frac{5}{4} - \frac{1}{4(m+1)}$$

$$\Updownarrow$$

$$\frac{1}{(m+1)^3} \le \frac{1}{4m(m+1)}$$

$$\Uparrow$$

$$4m \le (m+1)^2$$

法二：$m=1$ 时，$1 < \frac{5}{4}$

$m\geq 2$ $\sum_{j=1}^{m}\frac{1}{j^3}=1+\sum_{j=2}^{m}\frac{1}{j^3}<1+\sum_{j=2}^{m}\frac{1}{(j-1)(j+1)j}$

$=1+\frac{1}{2}\sum_{j=2}^{m}\left(\frac{1}{(j-1)j}-\frac{1}{j(j+1)}\right)=1+\frac{1}{2}\left(\frac{1}{2}-\frac{1}{m(m+1)}\right)<\frac{5}{4}$.

40 设 $S_i=a_1+a_2+\cdots+a_i$ $(i=1,2\cdots n)$

$\sum_{i=1}^{n}a_ib_i=a_1b_1+\sum_{i=2}^{n}a_ib_i=S_1b_1+\sum_{i=2}^{n}(S_i-S_{i-1})b_i$

$=\sum_{i=1}^{n}S_ib_i-\sum_{j=1}^{n-1}S_jb_{j+1}=S_nb_n+\sum_{i=1}^{n-1}(b_i-b_{i+1})S_i$

$i-1=j$ (阿贝尔恒等式)

45

设 $A_i=a_1+a_2+\cdots+a_i$ $(i=1,2\cdots n)$ $S=\sum_{k=1}^{n-1}\frac{a_k}{a_{k+1}}$，则

$S=\sum_{k=1}^{n-1}\frac{A_k-A_{k-1}}{a_{k+1}}=\sum_{k=1}^{n-1}\frac{A_k}{a_{k+1}}-\sum_{k=2}^{n-1}\frac{A_{k-1}}{a_{k+1}}=\sum_{k=1}^{n-1}\frac{A_k}{a_{k+1}}-\sum_{i=1}^{n-2}\frac{A_i}{a_{i+2}}$

定义 $A_0=0$ $=\sum_{k=1}^{n-2}\left(\frac{1}{a_{k+1}}-\frac{1}{a_{k+2}}\right)A_k+\frac{A_{n-1}}{a_n}$. $k-1=i$

$$\leq \sum_{k=1}^{n-2}\left(\frac{1}{a_{k+1}}-\frac{1}{a_{k+2}}\right)a_{k+1}+\frac{a_n}{a_n}=\sum_{k=1}^{n-2}\left(1-\frac{a_{k+1}}{a_{k+2}}\right)+1$$

$a_{k+2}\geq a_{k+1}$ $(k=1,2\dots n-2)$ $\frac{1}{a_{k+1}}-\frac{1}{a_{k+2}}\geq 0$

$$=n-1+\frac{a_1}{a_2}-\sum_{k=1}^{n-1}\frac{a_k}{a_{k+1}}$$

$$=n-1+\frac{a_1}{a_2}-S\leq n-1+1-S$$ $\therefore 2S\leq n$ $S\leq\frac{n}{2}$.

等号成立条件: $\frac{a_1}{a_2}=1$ $a_1=a_2$. $a_1+a_2=a_3$ $\cdots$ $a_1+a_2+\cdots+a_{n-1}=a_n$

此时假设 $a>0$ $a_1=a$. $a_2=a$. $a_3=2a$ $a_4=2^2a$ $\cdots$ $a_n=2^{n-2}a$

$\therefore S_{max}=\frac{n}{2}$

4. 设 $f(x,y,z)=(x^2-yz)(y^2-zx)(z^2-xy)$. 则 $f(x,0,z)=-x^3z^3$

当 x. z 一正一负时 $f(x,0,z)>0$ $\because f(-x,-y,-z)=f(x,y,z)$

不妨假设 $x+y+z\geq 0$

假设 x_0. y_0. z_0 满足 $x_0+y_0+z_0\geq 0$. $x_0^2+y_0^2+z_0^2=1$ 且 $f(x_0,y_0,z_0)$ 最大

不妨假设 $x_0 \geq y_0 \geq z_0$. ∵ $f(x_0,y_0,z_0)\geq 0$ ∴ $x_0>0$

$x_0^2-y_0z_0>0$ ∵ $f(x_0,y_0,z_0)-f(x_0,-y_0,-z_0)\geq 0$

$$=(x_0^2-y_0z_0)(y_0^2-z_0x_0)(z_0^2-x_0y_0)-(x_0^2-y_0z_0)(y_0^2+x_0z_0)(z_0^2+x_0y_0)$$

$$=-2x_0(x_0^2-y_0z_0)(y_0^3+z_0^3)$$ ∴ $y_0^3+z_0^3\leq 0$

4)

若 $z_0=0$. 则 $y_0^3\leq 0$, $0=z_0\leq y_0\leq 0$ ∴ $y_0=0$. $f(x_0,y_0,z_0)=0$ 矛盾

∴ $z_0<0$ ∴ $y_0^2-z_0x_0>0$ 由 $f(x_0,y_0,z_0)>0$ 可得 $z_0^2-x_0y_0>0$

由均值不等式可得 $f(x_0,y_0,z_0)\leq\left(\dfrac{x_0^2-y_0z_0+y_0^2-z_0x_0+z_0^2-x_0y_0}{3}\right)^3$

$$=\left(\frac{\frac{3}{2}(x_0^2+y_0^2+z_0^2)-\frac{1}{2}(x_0+y_0+y_0)^2}{3}\right)^3=\left(\frac{\frac{3}{2}-\frac{1}{2}(x_0+y_0+z_0)^2}{3}\right)^3$$

$\leq\left(\dfrac{\frac{3}{2}}{3}\right)^3=\dfrac{1}{8}$. 等号成立条件: $x_0^2-y_0z_0=y_0^2-z_0x_0=z_0^2-x_0y_0$

$x_0+y_0+z_0=0$. $x_0^2+y_0^2+z_0^2=1$. 有 $x_0=\frac{1}{2}$, $y_0=\frac{\sqrt{5}-1}{4}$, $z_0=-\frac{\sqrt{5}+1}{4}$

时等号成立 ∴ $f_{max}=\frac{1}{8}$.

42. 对于 $i=2,3\cdots2024$ 由均值不等式

$$x_i^i+\underbrace{\alpha_i+\cdots+\alpha_i}_{i-1}\ge i\sqrt[i]{x_i^i\underbrace{\alpha_i\cdots\alpha_i}_{i-1}}=i\cdot\alpha_i^{\frac{i-1}{i}}\cdot x_i$$

α_i 是待定的正数，令 $i\alpha_i^{\frac{i-1}{i}}=1$ ∴ $\alpha_i=i^{-\frac{i}{i-1}}$

则 $x_i^i\ge x_i-(i-1)\cdot i^{-\frac{i}{i-1}}$

等号成立时 $x_i^i=\alpha_i=i^{-\frac{i}{i-1}}$ 即 $x_i=i^{-\frac{1}{i-1}}<1$

$$\therefore \sum_{i=1}^{2024}x_i^i\ge\sum_{i=1}^{2024}x_i-\sum_{i=2}^{2024}(i-1)i^{-\frac{i}{i-1}}=2023-\sum_{i=2}^{2024}(i-1)i^{-\frac{i}{i-1}}.$$

等号成立的条件 $x_1=2023-\sum_{i=2}^{2024}i^{-\frac{1}{i-1}}>0$

43. 设 $\{a_1,a_2,a_3,a_4,a_5\}=T$，$|a_i-a_j|$ $(1\le i<j\le5)$ 要属于集合 a_1,a_2,a_3,a_4,a_5 中，没有两项相等，不妨假设 $a_1>a_2>\cdots>a_5$

要使f最大 一定有 $a_1=1$. $a_5=0$

$\therefore f(1,a_2,a_3,a_4,0)=(1-a_2)(1-a_3)(1-a_4)(a_2-a_3)(a_2-a_4)a_2(a_3-a_4)a_3a_4$

设 $a_2-a_4=x\in(0,1)$ 则

$f(1,a_2,a_3,a_4,0)=(1-a_2)a_4\cdot(1-a_3)a_3\cdot(1-a_4)a_2\cdot(a_2-a_3)(a_3-a_4)\cdot(a_2-a_4)$

$\leq\left(\frac{1-a_2+a_4}{2}\right)^2\cdot\left(\frac{1-a_3+a_3}{2}\right)^2\left(\frac{1-a_4+a_2}{2}\right)^2\left(\frac{a_2-a_3+a_3-a_4}{2}\right)^2\cdot(a_2-a_4)$

同时取到等号 当且仅当 $a_2+a_4=1$ $a_3=\frac{1}{2}$

$=\left(\frac{1-x}{2}\right)^2\cdot\frac{1}{4}\cdot\left(\frac{1+x}{2}\right)^2\left(\frac{x}{2}\right)^2x=\frac{1}{2^8}(x^7-2x^5+x^3)\triangleq g(x)$

$g'(x)=\frac{1}{2^8}(7x^6-10x^4+3x^2)$. 令 $g'(x)=0$ 可得 $\frac{1}{2^8}x^2(7x^2-3)(x^2-1)=0$

有唯一的实根 $x_0=\sqrt{\frac{3}{7}}=\frac{\sqrt{21}}{7}\in(0,1)$

且 $x\in(0,x_0)$ 时 $g(x)>0$. $x\in(x_0,1)$ 时 $g'(x)<0$

($g(x)$单调递增) ($g(x)$单调递减)

$\therefore$ $g(x)$的最大值为 $g(x_0)=\frac{3\sqrt{21}}{38416}$

由 $a_2-a_4=x_0=\frac{\sqrt{21}}{7}$ 可得 $a_2=\frac{7+\sqrt{21}}{14}$，$a_4=\frac{7-\sqrt{21}}{14}$