

Lexicon

A5_210mm x 148mm

30 sheets

数竞

2024.7 杭州

代数

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数列

$$1. S(nk) - S(n) - S(k)$$

$$= \sum_{t=2}^n \frac{1}{t} + \frac{1}{2} - \sum_{t=k+1}^{nk} \frac{1}{t}$$

$$= \frac{1}{2} + \sum_{t=2}^n \left(\frac{1}{t} - \frac{1}{tk(k-1)} - \frac{1}{tk(k-2)} - \dots - \frac{1}{tk} \right)$$

$$= \frac{1}{2} - \sum_{t=2}^n \left(\frac{k-1}{tk(tk-(k-1))} + \frac{k-2}{tk(tk-(k-2))} + \dots + \frac{1}{tk(tk-1)} \right)$$

$$> \frac{1}{2} - \sum_{t=2}^n \left(\frac{k-1}{t(t-1)k^2} + \frac{k-2}{t(t-2)k^2} + \dots + \frac{1}{t(t-1)k^2} \right)$$

$$> \frac{1}{2} - \sum_{t=2}^n \frac{1}{2t(t-1)}$$

$$= \frac{1}{2} - \frac{1}{2} \left(1 - \frac{1}{n} \right) = \frac{1}{2n} > 0.$$

$$2. \text{ 令 } a_k = a_1 + a_2 + \dots + a_{k-1}, k=2, 3, \dots, n$$

$$\text{便可得 } a_k = 2^{k-2} a_1, k=2, 3, \dots, n$$

$$\text{此时 } \frac{a_1}{a_2} + \frac{a_2}{a_3} + \dots + \frac{a_{n-1}}{a_n} \text{ 取 } \frac{n}{2}$$

$$\text{下证 } \frac{a_1}{a_2} + \frac{a_2}{a_3} + \dots + \frac{a_{n-1}}{a_n} \leq \frac{n}{2}$$

$$\text{令 } S_k = a_1 + a_2 + \dots + a_k, k=1, 2, 3, \dots, n$$

$$\text{则 } \sum_{k=1}^{n-1} \frac{a_k}{a_{k+1}} = \sum_{k=1}^{n-1} \frac{S_k - S_{k+1}}{a_{k+1}} = \sum_{k=1}^{n-1} \frac{S_k}{a_{k+1}} - \sum_{k=0}^{n-1} \frac{S_k}{a_{k+2}}$$

$$= \sum_{k=1}^{n-2} S_k \left(\frac{1}{a_{k+1}} - \frac{1}{a_{k+2}} \right) + \frac{S_{n-1}}{a_n} \quad (\text{阿贝尔变换})$$

$$\leq \sum_{k=1}^{n-2} a_{k+1} \left(\frac{1}{a_{k+1}} - \frac{1}{a_{k+2}} \right) + 1 \quad (\text{不能使用 } a_{k+2} \geq a_1 + a_2 + \dots + a_{k+1}, \text{ 题})$$

$$= n-1 + \frac{a_1}{a_2} - \sum_{k=1}^{n-1} \frac{a_k}{a_{k+1}}$$

$$\leq n-1+1 - \sum_{k=1}^{n-1} \frac{a_k}{a_{k+1}} = n - \sum_{k=1}^{n-1} \frac{a_k}{a_{k+1}} \quad (\text{移项即可})$$

$$3. \quad a(n) = a(a(n-1)) + a(n - a(n-1))$$

归纳证明 $1 \leq a(n) \leq n$

$n=1, 2$ 时显然.

设对 $n-1$ 时成立, $n \geq 3$, 则对 n :

$$a(n) = a(a(n-1)) + a(n - a(n-1)) \leq a(n-1) + (n - a(n-1)) = n$$

由归纳假设 $1 \leq n - a(n) \leq n-1$

首先证明一个引理: 对 $\forall n \in \mathbb{N}^*$, $a(n+1) - a(n) = 0$ 或 1

注意到 $a(1) = a(2) = 1$, $a(3) = 2$, $n=1, 2$ 时成立

设对不大于 $n-1$ ($n \geq 3$) 时成立, 对 n :

$$a(n+1) = a(a(n)) + a(n+1 - a(n))$$

$$a(n) = a(a(n-1)) + a(n - a(n-1))$$

若 $a(n) = a(n-1)$ 则 $a(a(n-1)) = a(a(n))$

$$a(n+1) - a(n) = a(n+1 - a(n-1)) - a(n - a(n-1))$$

由 $1 \leq n - a(n-1) \leq n-1$, 归纳假设知 $a(n+1) - a(n) = 0$ 或 1

若 $a(n) = a(n-1) + 1$ 则 $n+1 - a(n) = n - a(n-1)$

$$a(n+1) - a(n) = a(a(n-1) + 1) - a(a(n-1)) = 0 \text{ 或 } 1$$

由 $1 \leq a(n-1) \leq n-1$, 引理成立

反证, 若 M 是满足 $a(2M) > 2a(M)$ 的最小正整数

若 $a(M) = a(M-1) + 1$ 则

$$a(2M) > 2a(M) \geq 2a(M-1) + 2 \geq a(2M-2) + 2 \geq a(2M) \text{ 矛盾}$$

若 $a(M) = a(M-1)$, 记为 t

$$\begin{aligned}
 a(m) &= a(a(m-1)) + a(m - a(m-1)) \text{ 有 } t = a(t) + a(m-t) \\
 \text{由 } a(2m-2) &\leq 2t, a(2m) > 2t \text{ 由引理: } a(2m-1) = 2t \text{ 或 } 2t+1 \\
 \text{若 } a(2m-1) &= 2t \text{ 则 } a(2m) = a(a(2m-1)) + a(2m - a(2m-1)) \\
 &= a(2t) + a(2m-2t) \\
 &\leq 2a(t) + 2a(m-t) = 2t, \text{ 矛盾}
 \end{aligned}$$

若 $a(2m-1) = 2t+1$ 知 $a(2m-2) = 2t$

$$\begin{aligned}
 a(2m-1) &= a(a(2m-2)) + a(2m-1 - a(2m-2)) \\
 &= a(2t) + a(2m-1-2t) \\
 &= 2a(t) + a(2m-2t) \leq 2t, \text{ 矛盾}
 \end{aligned}$$

4. 结合(1)(2)要证结论, 想到与 $f(a+b)$ 、乘积有关的等式:

$$f^n(a+b) = f((a+b)^n)$$

推广(2), 对正整数 k 归纳证明 $f(a_1 + a_2 + \dots + a_{2^k})$

$$\leq 2^k \max \{f(a_1), f(a_2), \dots, f(a_{2^k})\}$$

$k=1$ 时显然, 设对于 k 成立, 对 $k+1$,

考虑 $2^{k-1} < n < 2^k$ 情形

$$\begin{aligned}
 \text{知 } f(a_1 + a_2 + \dots + a_n) &= 2 \max \{f(a_1 + a_2 + \dots + a_{2^{k-1}}), f(a_{2^{k-1}+1} + \dots + a_n)\} \\
 &\leq 2 \max \{2^{k-1} \max \{f(a_1), f(a_2), \dots, f(a_{2^{k-1}})\}, \\
 &\quad 2(n-2^{k-1}) \max \{f(a_{2^{k-1}+1}), f(a_{2^{k-1}+2}), \dots, f(a_n)\}\} \\
 &\leq \max \{2^k, 4(n-2^{k-1})\} \max \{f(a_1), f(a_2), \dots, f(a_n)\}
 \end{aligned}$$

$$\leq 2n \max \{f(a_1), f(a_2), \dots, f(a_n)\} \#$$

$$\text{从 } \forall n \in \mathbb{N}^*, f(a_1 + a_2 + \dots + a_n) \leq 2n \max \{f(a_1), f(a_2), \dots, f(a_n)\}$$

$$f(n) \leq 2n f(1), \forall n \in \mathbb{N}^*$$

$$f^n(a+b) = f((a+b)^n) = f\left(\sum_{i=0}^n C_n^i a^i b^{n-i}\right)$$

$$\leq 2(n+1) \max_{0 \leq i \leq n} \{f(C_n^i a^i b^{n-i})\}$$

$$\leq 2(n+1) \sum_{i=0}^n f(C_n^i) f(a^i) f(b^{n-i})$$

$$\leq 4(n+1) f(1) \sum_{i=0}^n C_n^i f(a)^i f(b)^{n-i}$$

$$= 4(n+1) f(1) (f(a) + f(b))^n$$

$$\text{故 } f(a+b) \leq \sqrt[n]{4(n+1)f(1)} (f(a) + f(b))$$

$$\text{令 } n \rightarrow +\infty \text{ 得 } f(a+b) \leq f(a) + f(b)$$

5. 不难算出 $a_1 = a_2 = a_3 = a_4 = 1, a_5 = a_6 = a_7 = 2$

$$\text{对 } k \text{ 归纳证明 } a_{2^k+k+1+2m-1} = a_{2^k+k+2+2m-1} = 2^k + m$$

$$m = 1, 2, \dots, 2^{k-1} - 1, a_{2^k+k} = a_{2^k+k+1} = a_{2^k+k+2} = 2^k$$

$k=1$ 时由 $a_5 = a_6 = a_7 = 2$ 知

设命题时对不大于 k 成立, 对 $k+1$, 由归纳假设

$$a_1 + a_2 + \dots + a_{2^k+k+2} = 2 \sum_{i=1}^{2^k} i + \sum_{i=1}^k 2^i + 1 = 2^{2k} + 3 \times 2^k$$

$$\text{对 } m = 1, 2, \dots, 2^k - 1, \text{ 归纳证明 } a_{2^k+k+2+2m-1}$$

$$= a_{2^k+k+2+2m} = 2^k + m, m = 1, 2, \dots, 2^k - 1, 2^k$$

$$m \geq 1 \text{ 时, } a_1 + a_2 + \dots + a_{2^k+k+2} = 2^{2k} + 3 \times 2^k$$

$$(2^k + 1)^2 < 2^{2k} + 3 \times 2^k < (2^k + 2)^2 \text{ 故 } a_{2^k+k+4} = 2^k + 1$$

设命题对不大于 m 成立, $1 \leq m \leq 2^n - 1$

对 $m+1$ 归纳假设

$$a_1 + a_2 + \dots + a_{2^k + k + 2 + 2m} = 2^{2k} + 3 \times 2^k + 2((2^k + 1) + (2^k + 2) + \dots + (2^k + m)) = (2^k + m + 1)^2 + 2^k - m - 1$$

$$\text{故 } a_{2^k + k + 2 + 2m + 1} = 2^k + m + 1$$

$$a_1 + a_2 + \dots + a_{2^k + k + 2 + 2m + 1} = (2^k + m + 1)^2 + 2^{k+1}$$

$$\text{故 } a_1 + a_2 + \dots + a_{2^k + k + 2 + 2m + 1} = (2^k + m + 1)^2 + 2^{k+1} \quad \#$$

$$\text{此时 } a_1 + a_2 + \dots + a_{2^{k+1} + k + 2} = 2^{2k} + 3 \times 2^k + 2((2^k + 1) + (2^k + 2) + \dots + (2^k + 2^k)) = 2^{2k+2} + 2^{k+2} = (2^{k+1} + 1)^2 - 1$$

$$\text{故 } a_{2^{k+1} + k + 3} = 2^{k+1} \quad \#$$

出现超过 2 次的数为 2^k , ($k \geq 0$)

6. 法一: 变形

$$((a_n - a_{n-1}) + (a_n - a_{n-2}))^2 = (a_n - a_{n-1})^2 - 2(a_n - a_{n-1})(a_n - a_{n-2}) + (a_n - a_{n-2})^2$$

$$\text{即 } (a_n - a_{n-2})^2 = (a_n - a_{n-1})^2 - 2(a_n - a_{n-1})(a_n - a_{n-2}) + (a_n - a_{n-2})^2$$

$$\text{类似地 } (b_n - b_{n-2})^2 = (b_n - b_{n-1})^2 - 2(b_n - b_{n-1})(b_n - b_{n-2}) + (b_n - b_{n-2})^2$$

$$\text{求和 } (a_n - a_{n-2})^2 + (b_n - b_{n-2})^2 + (a_{n-1} - a_{n-1})^2 + (b_n - b_{n-2})^2$$

$$\geq (a_n - a_{n-1})^2 + (b_n - b_{n-1})^2$$

知 $(a_n - a_{n-1})^2 + (b_n - b_{n-1})^2$ 为单调不增的非负整数数列

$\exists N, n \geq N$ 时, $(a_n - a_{n-1})^2 + (b_n - b_{n-1})^2$ 为定值 M

$n \geq N+1$ 时 $M = M + (a_n - a_{n-2})^2 + (b_n - b_{n-2})^2$

$$\therefore a_n = a_{n-2}, b_n = b_{n-2}$$

$$\therefore \exists k \text{ 使 } a_k = a_{k+1000}$$

法二：图形

$$\text{设 } P_n(a_n, b_n), \text{ 知 } \overrightarrow{P_{n-1}P_n} = (a_n - a_{n-1}, b_n - b_{n-1})$$

$$P_{n-2} \quad P_n \quad \overrightarrow{P_{n-2}P_n} = (a_n - a_{n-2}, b_n - b_{n-2}), \overrightarrow{P_{n-2}P_n} \cdot \overrightarrow{P_{n-1}P_n} = 0$$

$$\text{勾股定理 } |P_{n-1}P_{n-2}|^2 = |P_nP_{n-1}|^2 + |P_{n-2}P_{n-1}|^2$$

$$\geq |P_nP_{n-1}|^2$$

(构成单调不减的非负整数列)

$$\therefore P_n P_{n+2} = 0 \quad \#$$

7. S_n 表示对数 a_i 不同选择, X_n 可以取到所有可能值的集合

$$\text{例如 } S_3 = \left\{ \frac{1}{4}, \frac{7}{5}, \frac{3}{5}, \frac{3}{4}, \frac{4}{3}, \frac{5}{3}, \frac{5}{2}, 4 \right\}$$

$$\text{由 } a_{n+1} = -1 \text{ 时, } X_{n+1} = \frac{1}{X_n + 1}, a_{n+1} = 1 \text{ 时 } X_{n+1} = X_n + 1$$

$$\text{知 } S_{n+1} = \left\{ \frac{1}{x+1}, x+1 \mid x \in S_n \right\}$$

$$|S_{n+1}| = 2|S_n|, |S_n| = 2^n$$

$$\text{设 } S_n \text{ 中元素从小到大为 } a_1^{(n)} < a_2^{(n)} < \dots < a_{2^n}^{(n)}$$

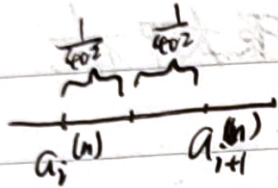
$$\text{知 } S_{n+1} = \left\{ \frac{1}{a_{2^n}^{(n)} + 1}, \frac{1}{a_{2^n-1}^{(n)} + 1}, \dots, \frac{1}{a_2^{(n)} + 1}, a_1^{(n)} + 1, a_2^{(n)} + 1 \right\}$$

$$\text{且 } \frac{1}{a_{2^n}^{(n)} + 1} < \frac{1}{a_{2^n-1}^{(n)} + 1} < \dots < \frac{1}{a_2^{(n)} + 1} < a_1^{(n)} + 1 < \dots < a_{2^n}^{(n)} + 1$$

$$\text{故 } a_{\bar{i}}^{(n+1)} = \frac{1}{a_{2^n+1-\bar{i}}^{(n)} + 1}, \bar{i} = 1, 2, \dots, 2^n \quad \textcircled{1}$$

$$a_{\bar{i}}^{(n+1)} = a_{\bar{i}-2^n}^{(n)} + 1, \bar{i} = 2^n + 1, 2^n + 2, \dots, 2^{n+1}$$

由此知 $a_{2^{n+1}-i}(n+1) = a_{2^{n+1}-i}(n) + 1 = \frac{1}{a_i(n+1)}$, $i=1, 2, \dots, 2^n$
 即有 $a_{2^{n+1}-i}(n) = \frac{1}{a_i(n)}$, $i=1, 2, \dots, 2^{n-1}, 2^{n-1}+1, \dots, 2^n$ ②,
 由①②式, 有 $a_i(n+1) = \frac{a_i(n)}{a_i(n)+1}$, $i=1, 2, \dots, 2^n$ ③



$$\frac{1}{2^{n+1}} = \frac{1}{2^{n+1}}$$

下面归纳证明: $n \geq 2$ 时, $a_{i+1}(n) - a_i(n) \leq 2^{n-1} - 1$

$n=2$ 显然成立

设对 n 成立, 对 $n+1$:

$i \leq 2^{n-1} - 1$ 时, 由③式: $a_i(n+1) = \frac{a_i(n)}{a_i(n)+1}$, 归纳可证 $a_i(n) = \frac{1}{n+1}$

$$\begin{aligned} \text{再由③式知 } a_{i+1}(n+1) - a_i(n+1) &= \frac{a_{i+1}(n) - a_i(n)}{(1+a_{i+1}(n))(1+a_i(n))} \\ &\leq \frac{1}{(1+\frac{1}{n+1})^2} \leq \frac{1}{1+\frac{2}{n+1}} \leq \frac{1}{2n+1} \end{aligned}$$

$i=2^{n-1}$ 时, 知 $a_{i+1}(n) = \frac{1}{n}$

$$\text{由①知 } a_{2^{n-1}}(n) = \frac{1}{a_{i+1}(n)+1} = \frac{n}{n+1}$$

类似地, 由②③知 $a_{2^{n-1}+1}(n) = \frac{n+1}{n}$

$$a_{2^{n-1}}(n+1) = \frac{n}{2n+1}, \quad a_{2^{n-1}+1}(n+1) = \frac{n+1}{2n+1}$$

$$\text{从而 } a_{2^{n-1}+1}(n+1) - a_{2^{n-1}}(n+1) = \frac{1}{2n+1} \quad \#$$

$2^{n-1}+1 \leq i \leq 2^n - 1$ 时

$$\text{显然 } a_i(n+1) = 1 - a_{2^{n+1}-i}(n+1)$$

$$a_{i+1}(n+1) = 1 - a_{2^{n+1}-i-1}(n+1)$$

$$\text{故 } a_{i+1}(n+1) - a_i(n+1) = a_{2^{i+1}-1}(n+1) - a_{2^i-1}(n) \leq \frac{1}{2^{n+1}}$$

回到原题, 设 $b_i = a_i^{(10^i)}$, $i=1, 2, \dots, 2^{10^1}$

$$b_i \leq x < b_{i+1} \text{ 时, } |x - b_i| + |x - b_{i+1}| = b_{i+1} - b_i \leq \frac{1}{2^{2^i}}$$

故 $|x - b_i|$, $|x - b_{i+1}|$ 中有一项大于 $\frac{1}{4 \cdot 2^i}$

8. $\lambda < \frac{1}{2}$ 时, 取 $\alpha > 1$, 使 $\alpha + \frac{1}{\alpha} < \frac{1}{\lambda}$

$$\text{令 } a_n = \frac{1}{\alpha + \frac{1}{\alpha}}, b_n = \alpha^n, \{b_n\} \text{ 无界}$$

下证: $\lambda = \frac{1}{2}$ 时满足题设. $\alpha_0 > 1$

(注意 $b_n + b_{n+2}$ 符号不确定, 比如 $b_n = \cos \frac{n\pi}{t}$, 不能 $b_{n+1} > \lambda(b_n b_{n+2})$)

$$* b_{n+1}(b_{n+2} - b_n) = a_n(b_{n+2}^2 - b_n^2)$$

$$\text{思路: 从 } 0 \text{ 叠加到 } n-2: \sum_{i=0}^{n-2} a_i(b_{i+2}^2 - b_i^2) = b_{n-1}b_n - b_0b_1$$

$$\text{又 } \sum_{i=0}^{n-2} (b_{i+2}^2 - b_i^2) = b_{n-2}^2 + b_n^2 - b_0^2 - b_1^2$$

$$\text{由以上两式 } \sum_{i=0}^{n-2} (2a_i - 1)(b_{i+2}^2 - b_i^2) = (b_0 - b_1)^2 - (b_{n-1} - b_n)^2$$

$$(2a_{n-2} - 1)b_n^2 + (2a_{n-3} - 1)b_{n-1}^2 - \underbrace{2(a_{n-2} - a_{n-4})b_{n-2}^2}_{\text{不好处理}}$$

再想变形

$$\text{思路中实际是 } (2a_n - 1)(b_{n+2}^2 - b_n^2) = (b_n - b_{n+1})^2 - (b_{n+1} - b_{n+2})^2$$

求和

除过去

$$* b_{n+2} - b_n = \frac{1}{2a_{n-1}} ((b_n - b_{n+1})^2 - (b_{n+1} - b_{n+2})^2)$$

$$\text{从 } 0 \text{ 到 } n-2 \text{ 叠加得: } b_n^2 + b_{n-1}^2 - b_0^2 - b_1^2 = \frac{1}{2a_{n-2}-1} (b_{n-1} - b_n)^2 + \sum_{i=1}^{n-2} \left(\frac{1}{2a_{i-1}} - \frac{1}{2a_{i-2}} \right) (b_i - b_{i+1})^2 + \frac{1}{2a_0-1} (b_0 - b_1)^2$$

$$\leq \frac{1}{2a_0-1} (b_0 - b_1)^2$$

$$\text{and } b_n^2 \leq b_0^2 + b_1^2 + \frac{1}{2a_0-1} (b_0 - b_1)^2 \quad \#$$

n元不等式

1. 令 $a_1 = a_2 = \dots = a_{n-1} = 1, a_n = x$

$$\text{得 } \frac{n-1}{n-2+x} + \frac{kx}{n-1} \geq \frac{n-1}{n-2}$$

$$\frac{kx}{n-1} \geq \frac{(n-1)x}{(n-2)(n-2+x)}, \text{ 即 } k \geq \frac{(n-1)^2}{(n-2)(n-2+x)}$$

$$\text{令 } x \rightarrow 0, k \geq \frac{(n-1)^2}{(n-2)(n-2)} = \left(\frac{n-1}{n-2}\right)^2$$

$$\text{下证 } \sum_{i=1}^{n-1} \frac{a_i}{s-a_i} + \left(\frac{n-1}{n-2}\right)^2 \frac{a_n}{s-a_n} \geq \frac{n-1}{n-2}$$

$$\text{上式式等价于 } \sum_{i=1}^{n-1} \frac{s}{s-a_i} + \left(\frac{n-1}{n-2}\right)^2 \frac{a_n}{s-a_n} \geq \frac{(n-1)^2}{n-2}$$

$$\text{由柯西不等式 } \sum_{i=1}^{n-1} \frac{s}{s-a_i} \geq s \cdot \frac{(n-1)^2}{\sum_{i=1}^{n-1} (s-a_i)} = \frac{(n-1)^2 s}{(n-2)s + a_n}$$

$$(\text{注意柯西不等式常见变形 } \sum_{i=1}^n \frac{a_i^2}{b_i} \geq \frac{(\sum_{i=1}^n a_i)^2}{\sum_{i=1}^n b_i})$$

$$\text{即证 } \frac{s}{(n-2)s + a_n} + \frac{a_n}{(n-2)^2(s-a_n)} \geq \frac{1}{n-2}$$

$$\text{即 } \frac{1}{(n-2)^2} \cdot \frac{a_n}{s-a_n} \geq \frac{a_n}{(n-2)(n-2)s + a_n} \quad \#$$

2. 若 $(a, b) \notin T$, 知 $\sum_{(i,j) \in T} x_i x_j$ 形如 $x_a(x_0 + \dots) + x_b(x_0 + \dots + x_n)$

考虑任一组满足 $x_i = 1, x_i$ 非负的 $x_1, x_2, \dots, x_n$

$$\text{若 } (a, b) \notin T \text{ 知 } \sum_{(i,j) \in T} x_i x_j = x_a \sum_{\substack{(i,j) \in T \\ i \neq a, j \neq a}} x_j + x_b \sum_{\substack{(i,j) \in T \\ i \neq b, j \neq b}} x_j$$

$$+ \sum_{\substack{(i,j) \in T \\ i, j \text{ 均不为 } a, b}} x_i x_j$$

若 x_a 系数较小, 知 x_a 变为 0, x_b 变为 $x_a + x_b$, 和式不减

若 x_a 系数不小于 x_b 系数, x_a 变为 $x_a + x_b$, x_b 变为 0, 和式不减
这么调整下去, 每次调整 $x_1, x_2, \dots, x_n$ 中 0 数目至少加 1

有限次调整结束, 此时 $\forall (a, b) \notin T$, x_a 与 x_b 中至少有一个
 设 $\{i \mid x_i \neq 0\} = \{i_1, i_2, \dots, i_s\}$, $i_1 < i_2 < \dots < i_s$

知 $i_1 \mid i_2, i_2 \mid i_3, \dots, i_{s-1} \mid i_s$

$$i_2 \geq 2i_1, i_3 \geq 2i_2, \dots, i_s \geq 2i_{s-1}$$

$$i_s \geq 2^{s-1}, s-1 \leq \log_2 n, s \leq \lfloor \log_2 n \rfloor + 1$$

$$\begin{aligned} \sum_{(i,j) \in T} x_i x_j &= \sum_{1 \leq i_1 < i_2 \leq s} x_{i_1} x_{i_2} = \frac{1}{2} \left(\left(\sum_{i=1}^s x_{i_i} \right)^2 - \sum_{i=1}^s x_{i_i}^2 \right) \\ &= \frac{1}{2} - \frac{1}{2} \sum_{i=1}^s x_{i_i}^2 \\ &\leq \frac{1}{2} - \frac{1}{2s} \quad (\text{柯西不等式}) \\ &= \frac{\lfloor \log_2 n \rfloor}{2(\lfloor \log_2 n \rfloor + 1)} \end{aligned}$$

故不超过 $\frac{\lfloor \log_2 n \rfloor}{2(\lfloor \log_2 n \rfloor + 1)}$, 设 $k = \lfloor \log_2 n \rfloor$

取 $x_1 = x_2 = \dots = x_{2^k} = \frac{1}{k+1}$, 其余为 0 即可.

3. 试出 $n=5$ 满足题设

构造方式很多, 比如: $a_1 = 2x, a_2 = a_3 = 1, a_4 = a_5 = -(x+1)$

知 $a_1 + a_2 + a_3 + a_4 + a_5 = 0$

考虑 3 次, 5 次的等号, 令 $x = 1.5$

再微调至 $a_1 = 2.9, a_2 = a_3 = 1, a_4 = a_5 = -2.4$

使 $a_1 + a_2 + a_3 + a_4 + a_5 > 0$

再比如: $a_1 = 8, a_2 = 4, a_3 = 3, a_4 = -7, a_5 = -7$

下面考虑 $n=4$, 若 a_1, a_2, a_3, a_4 满足

不妨设 $a_1 \geq a_2 \geq a_3 \geq a_4$, 若其中恰一个正数

知 $0 > a_2 \geq a_3 \geq a_4$ 则 $a_1 > -(a_2 + a_3 + a_4)$

$$a_1^3 + a_2^3 + a_3^3 + a_4^3 > -(a_2 + a_3 + a_4)^3 + a_2^3 + a_3^3 + a_4^3 > 0, \text{矛盾}$$

若恰3个正数, 则 $|a_4|^3 > |a_1|^3 + |a_2|^3 + |a_3|^3$

$$|a_4| > |a_1|, |a_2|, |a_3|$$

$$|a_4|^5 > |a_1|^5 + |a_2|^5 + |a_3|^5, \text{矛盾}$$

故恰2个正数

$$\text{此时 } a_1 + a_2 > |a_3| + |a_4|$$

$$a_1^3 + a_2^3 < |a_3|^3 + |a_4|^3$$

$$a_1^5 + a_2^5 > |a_3|^5 + |a_4|^5$$

设 $x_i = |a_i|$, 知 $x_1 \geq x_2, x_4 \geq x_3$, 先给 x_1, x_2, x_3, x_4 排序

$$\text{知 } x_1^2 - x_1 x_2 + x_2^2 < x_3^2 - x_3 x_4 + x_4^2$$

$$\text{及 } x_1^2 + 2x_1 x_2 + x_2^2 > x_3^2 + 2x_3 x_4 + x_4^2$$

$$\text{即 } x_1 - x_2 < x_4 - x_3$$

$$\text{又 } x_1 + x_2 > x_3 + x_4$$

$$\text{故 } x_2 > x_3, \text{ 又 } x_1^3 + x_2^3 < x_3^3 + x_4^3 \text{ 知 } x_4 > x_1$$

故 $x_4 > x_1 \geq x_2 > x_3$, 最大元素为 x_4

$$\text{猜测 } x_1^5 + x_2^5 < x_3^5 + x_4^5$$

$$\text{只需证: } \frac{x_4^5 - x_3^5}{x_4^3 - x_3^3} > \frac{x_2^5 - x_1^5}{x_2^3 - x_1^3}$$

注意到对 $f(x) = x^{\frac{5}{3}}$, $x \in (0, +\infty)$

$$f'(x) = \frac{5}{3} x^{\frac{2}{3}} \nearrow$$

$$f(y) = \frac{f(y) - f(2)}{y - 2} > f'(2) \text{ 其中 } y > 2 > 0$$

$$\text{故 } \frac{x_2^5 - x_1^5}{x_2^2 - x_1^2} > \frac{5}{3} x_1^2 \geq \frac{5}{3} x_2^2 > \frac{x_2^5 - x_3^5}{x_2^2 - x_3^2} \text{ 矛盾}$$

$$\downarrow \frac{5}{3} (x_1^3)^{\frac{2}{3}}$$

故 $n=4$ 时不存在

$n \leq 3$ 时, 补 0 变成 $n=4$ 情形, 故 n 最小值为 5

$$4. \left(\text{考虑 } \sum_{i=1}^n \frac{a_i b_i c_i}{b_i c_i + c_i a_i + a_i b_i} = \sum_{i=1}^n \frac{a_i b_i c_i}{a_i (b_i + c_i) + b_i c_i} \right.$$

$$= \sum_{i=1}^n \frac{a_i \frac{b_i c_i}{b_i + c_i}}{a_i + \frac{b_i c_i}{b_i + c_i}} \quad \text{令 } x_i = \frac{b_i c_i}{b_i + c_i}$$

$$\text{引理: } a_i, b_i > 0, \text{ 则 } \sum \frac{a_i b_i}{a_i + b_i} \leq \sum (a_i + b_i) \leq \sum a_i \cdot \sum b_i$$

$$\text{证明: 知 } \sum \frac{a_i b_i}{a_i + b_i} = \sum \left(a_i - \frac{a_i^2}{a_i + b_i} \right) = \sum a_i - \sum \frac{a_i^2}{a_i + b_i}$$

$$\text{由柯西不等式: } \sum \frac{a_i^2}{a_i + b_i} \geq \frac{(\sum a_i)^2}{\sum (a_i + b_i)}$$

$$\text{从而 } \sum \frac{a_i b_i}{a_i + b_i} \leq \sum (a_i + b_i) \leq \left(\sum a_i - \frac{(\sum a_i)^2}{\sum (a_i + b_i)} \right) \sum (a_i + b_i)$$

$$= \sum a_i (\sum a_i + \sum b_i) - (\sum a_i)^2$$

$$= \sum a_i \cdot \sum b_i \quad \#$$

想办法各项均用引理放缩

$$\text{由引理 } \sum_{i=1}^n \frac{a_i b_i c_i}{b_i c_i + c_i a_i + a_i b_i} = \sum_{i=1}^n \frac{a_i x_i}{a_i + x_i} \leq \frac{\sum a_i \cdot \sum x_i}{\sum_{i=1}^n (a_i + x_i)}$$

$$\text{其中 } x_i = \frac{b_i c_i}{b_i + c_i}$$

$$\text{又 } \sum x_i \leq \frac{\sum b_i \cdot \sum c_i}{\sum (b_i + c_i)}$$

$$\text{从而 } \sum_{i=1}^n \frac{a_i b_i c_i}{b_i c_i + c_i a_i + a_i b_i} \leq \frac{\sum a_i \cdot \frac{\sum b_i \cdot \sum c_i}{\sum (b_i + c_i)}}{\sum a_i + \frac{\sum b_i \cdot \sum c_i}{\sum (b_i + c_i)}} = \frac{\sum a_i \cdot \sum b_i \cdot \sum c_i}{\sum a_i \cdot \sum b_i + \sum b_i \cdot \sum c_i + \sum c_i \cdot \sum a_i}$$

$$\begin{aligned} \alpha \sum_{i=1}^n \frac{a_i b_i + b_i c_i}{a_i + b_i + c_i} &= \sum_{i=1}^n \frac{b_i (a_i + c_i)}{b_i + (a_i + c_i)} \leq \frac{\sum b_i \sum (a_i + c_i)}{\sum b_i + \sum (a_i + c_i)} \\ &= \frac{\sum a_i \sum b_i + \sum b_i \sum c_i}{\sum a_i + \sum b_i + \sum c_i} \quad \text{由引理} \end{aligned}$$

$$\text{类似地 } \sum_{i=1}^n \frac{a_i c_i + c_i b_i}{a_i + b_i + c_i} \leq \frac{\sum a_i \sum c_i + \sum c_i \sum b_i}{\sum a_i + \sum b_i + \sum c_i}$$

$$\sum_{i=1}^n \frac{b_i a_i + a_i c_i}{a_i + b_i + c_i} \leq \frac{\sum b_i \sum a_i + \sum a_i \sum c_i}{\sum a_i + \sum b_i + \sum c_i}$$

$$\text{由引理, } \sum_{i=1}^n \frac{b_i c_i + c_i a_i + a_i b_i}{a_i + b_i + c_i} \leq \frac{\sum b_i \sum c_i + \sum b_i \sum a_i + \sum a_i \sum c_i}{\sum a_i + \sum b_i + \sum c_i}$$

$$5. P(x) = (x - z_1)(x - z_2) \cdots (x - z_n)$$

$$\text{故 } |P(z_0)| = P(z_0) \overline{P(z_0)}$$

$$= \prod_{j=1}^n (z_0 - z_j)(\overline{z_0} - \overline{z_j})$$

$$= \prod_{j=1}^n (|z_0|^2 - z_0 \overline{z_j} - \overline{z_0} z_j + |z_j|^2) \quad \rightarrow \text{正实数}$$

$$\leq \left(\frac{1}{n} \sum_{j=1}^n (|z_0|^2 - z_0 \overline{z_j} - \overline{z_0} z_j + |z_j|^2) \right)^n$$

$$\text{而 } \frac{1}{n} \sum_{j=1}^n (|z_0|^2 - z_0 \overline{z_j} - \overline{z_0} z_j + |z_j|^2)$$

$$= |z_0|^2 - z_0 \overline{\alpha} - \overline{z_0} \alpha + \beta^2$$

$$= (z_0 - \alpha)(\overline{z_0} - \overline{\alpha}) + \beta^2 - |\alpha|^2$$

$$= |z_0 - \alpha|^2 + \beta^2 - |\alpha|^2 < 1. \quad \#$$

$$6. \text{ 估计 } \sqrt{\frac{a_i^2 + 1}{2}} - \sqrt{a_i} = \frac{\frac{a_i^2 + 1}{2} - a_i}{\sqrt{\frac{a_i^2 + 1}{2}} + \sqrt{a_i}} = \frac{1}{2} \cdot \frac{(a_i - 1)^2}{\sqrt{\frac{a_i^2 + 1}{2}} + \sqrt{a_i}} \geq \frac{(a_i - 1)^2}{\sqrt{2(\frac{a_i^2 + 1}{2} + a_i)}} = \frac{(a_i - 1)^2}{a_i + 1}$$

$$\text{故 } \sum_{i=1}^n (\sqrt{\frac{a_i+1}{2}} - \sqrt{a_i}) \geq \frac{1}{2} \sum_{i=1}^n \frac{(a_i-1)^2}{a_i+1} = \frac{1}{2} \left(\sum_{i=1}^n (a_i-3) + \sum_{i=1}^n \frac{4}{a_i+1} \right) \\ = \frac{1}{2} \sum_{i=1}^n a_i - \frac{3}{2}n$$

$$\text{只需证: } \frac{1}{2} \sum_{i=1}^n a_i \geq 2 \sum_{i=1}^n \sqrt{a_i} - \frac{3}{2}n$$

注意 $\sum_{i=1}^n \frac{a_i}{a_i+1} = \frac{n}{2}$, 从 $\frac{a_i}{a_i+1}$ 中可放缩出 a_i

$$\frac{2a_i}{a_i+1} + \frac{a_i+1}{2} \geq 2\sqrt{a_i}$$

$$\text{从 1 加到 } n \text{ 得 } \frac{1}{2} \sum_{i=1}^n a_i + \frac{3}{2}n \geq 2 \sum_{i=1}^n \sqrt{a_i} \quad \#$$

7. 猜最大值, 下标大的项小, 等差数列, $a_n = 0$

$$a_i = n-i, \text{ 则 } c = \frac{\sum_{i=0}^n i(n-i)^2}{\sum_{i=0}^n (n-i)^2} = \frac{n^2-n}{4n+2}$$

下证其最大, 设 $c_0 = \frac{n^2-n}{4n+2}$, 我们证明 $\sum_{i=0}^n (i-c_0)a_i^2 \geq 0$

$$\text{知 } 0 \leq i < j \leq n \text{ 时 } \frac{a_j - a_i}{j-i} > \frac{a_n - a_i}{n-i}$$

$$a_j \geq \frac{n-j}{n-i} a_i + \frac{i-j}{n-i} a_n \geq \frac{n-j}{n-i} a_i$$

$$\frac{a_j}{n-j} \geq \frac{a_i}{n-i} \text{ 固定 } i$$

将 a_i 与等差数列 b_i 相比较

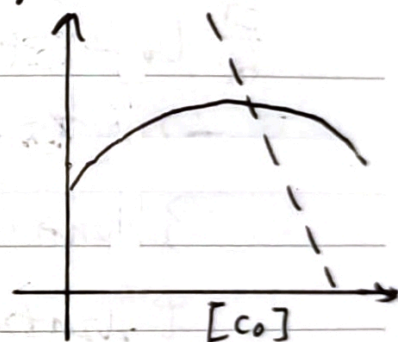
$$b_n = 0, b[c_0] = a[c_0], \text{ 且 } b_i = \frac{n-i}{n-[c_0]} a[c_0]$$

$$i \leq [c_0] < i < n \text{ 时, } \frac{a_i}{n-i} \geq \frac{a[c_0]}{n-[c_0]} = \frac{b[c_0]}{n-[c_0]} = \frac{b_i}{n-i}$$

故 $a_i \geq b_i$, 不知 $i=n$ 时成立

$$\text{故总有: } (i-c_0)a_i^2 \geq (i-c_0)b_i^2$$

$$\sum_{i=0}^n (i-c_0)a_i^2 \geq \sum_{i=0}^n (i-c_0)b_i^2 = 0$$



8. $x_n^3 x_{n+2}$

一方面, $x_n = \frac{3}{4}, x_{n+1} = \frac{1}{4}$, 其余各数均为0时取到 $\frac{27}{256}$

另一方面, 对 n 归纳证明, 对一切不小于2的正整数 n

及所有非负实数 $x_1, x_2, \dots, x_{2n}$

$$(x_n^2 + x_n x_{n+1} + x_{n-2} x_{n+2} + \dots + x_1 x_{2n-1})(x_{n+1}^2 + x_n x_{n+2} + \dots + x_2 x_{2n}) + (x_{n-1}^2 x_{n+2}^2 + x_{n-2}^2 x_{n+1}^2 + \dots + x_1^2 x_{2n}^2) \leq \frac{27}{256} (x_1 + x_2 + \dots + x_{2n})^4$$

假设命题对 n 成立, 对 $n+1$, 我们证对任意非负实数 $x_1, x_2, \dots, x_n$

$$(x_{n+1}^2 + x_n x_{n+2} + \dots + x_1 x_{2n+1})(x_{n+2}^2 + x_{n+1} x_{n+3} + \dots + x_2 x_{2n+2}) + (x_n^2 x_{n+3}^2 + x_{n-1}^2 x_{n+4}^2 + \dots + x_1^2 x_{2n+2}^2) \leq \frac{27}{256} (x_1 + x_2 + \dots + x_{2n+2})^4$$

右式减去左式关于 x_1 导数为

$$\begin{aligned} & \frac{27}{64} (x_1 + x_2 + \dots + x_{2n+2})^3 - x_{2n+1} (x_{n+2}^2 + x_{n+1} x_{n+3} + \dots + x_2 x_{2n+2}) \\ & - 2x_1 x_{2n+2}^2 \geq \frac{27}{64} (3x_{n+1}^2 (x_1 + x_2 + \dots + x_{2n} + x_{2n+2}) + \\ & 3x_{n+1} (x_1 + x_2 + \dots + x_{2n} + x_{2n+2})^2 + (x_1 + x_2 + \dots + x_{2n} + x_{2n+2})^3 - x_{2n+1} (x_{n+2}^2 - x_{n+1} x_{n+3} + \dots + x_2 x_{2n+2}) \\ & - 2x_1 x_{2n+2}^2 \geq \frac{27}{64} (3x_{n+1}^2 x_3 + 3x_{n+1} x_{n+2}^2 + x_{n+1} x_{n+3} + \dots + x_2 x_{2n+2}) - 2x_1 x_{2n+2}^2 \geq 0 \end{aligned}$$

故只需证 $x_1 = 0$ 的情形, 同理 $x_{2n+2} = 0$ 即 $x_1 = x_{2n+2} = 0$

$$\text{令 } y_i = x_{i+1}, \text{ 要证: } (y_n^2 + y_{n-1} y_{n+1} + \dots + y_1 y_{2n-1})(y_{n+1}^2 + y_n y_{n+2} + \dots + y_2 y_{2n}) + (y_{n-1}^2 y_{n+2}^2 + y_{n-2}^2 y_{n+3}^2 + \dots + y_1^2 y_{2n}^2) \leq \frac{27}{256} (y_1 + y_2 + \dots + y_{2n})^4, \text{ 由归纳假设, 并}$$

下证 $n=2$ 时命题成立, 即证:

$$\begin{aligned}
 & x_2^2 x_3^2 + x_1^2 x_4^2 + x_1 x_2 x_3 x_4 + x_1 x_3^3 + x_2^3 x_4 \leq \frac{27}{256} (x_1 + x_2 + x_3 + x_4)^4 \\
 & \frac{27}{256} (x_1 + x_2 + x_3 + x_4)^4 \geq \frac{27}{256} [(x_1 + x_3)^4 + (x_2 + x_4)^4 + (4x_2^3 x_3 + 6x_2^2 x_3^2 + 4x_2 x_3^3) \\
 & \quad + (4x_1^3 x_4 + 6x_1^2 x_4^2 + 4x_1 x_4^3) + 24x_1 x_2 x_3 x_4] \\
 & \Rightarrow \frac{27}{256} (4x_2^3 x_3 + 6x_2^2 x_3^2 + 4x_2 x_3^3) \geq \frac{27}{256} \cdot 14x_2^2 x_3^2 \geq x_2^2 x_3^2 \\
 & \quad \frac{27}{256} (4x_1^3 x_4 + 6x_1^2 x_4^2 + 4x_1 x_4^3) \geq x_1^2 x_4^2 \\
 & \quad \frac{27}{256} (x_1 + x_3)^4 \geq x_1 x_3^2 \text{ (均值不等式)}, \quad \frac{27}{256} (x_2 + x_4)^4 \geq x_2 x_4^3 \#
 \end{aligned}$$

数列不等式

1. 柯西不等式放掉平方, 另一边要出现 $\sum_{i=1}^n \frac{1}{a_i}$

$$\text{即 } \left(\sum_{k=1}^n \frac{1}{a_k^2 + x^2} \right)^2 \leq \sum_{k=1}^n \frac{1}{a_k} \cdot \sum_{k=1}^n \frac{a_k}{(a_k^2 + x^2)^2} \leq \sum_{k=1}^n \frac{a_k}{(a_k + x^2)^2}$$

$$\text{要证 } \sum_{k=1}^n \frac{a_k}{(a_k^2 + x^2)^2} \leq \frac{1}{2} \cdot \frac{1}{a_1(a_1+1) + x^2}$$

$$\text{只需证: } \frac{a_k}{(a_k^2 + x^2)^2} \leq \frac{1}{2} \left(\frac{1}{a_k(a_k+1) + x^2} - \frac{1}{a_{k+1}(a_{k+1}+1) + x^2} \right)$$

$$\text{而不等式右边} \geq \frac{1}{2} \left(\frac{1}{a_k(a_k+1) + x^2} - \frac{1}{a_k(a_{k+1}) + x^2} \right)$$

$$= \frac{1}{2} \cdot \frac{2a_k}{(a_k^2 - a_k + x^2)(a_k^2 + a_{k+1} + x^2)}$$

$$> \frac{a_k}{(a_k^2 + x^2)^2} \quad \#$$

2. 对 $1 \leq i \leq k$, $0 < \varepsilon < \frac{1}{2i^2}$

$$\text{取 } x_1 = \frac{1}{i} - \varepsilon, x_2 = \frac{1}{i} - 2\varepsilon, \dots, x_i = \frac{1}{i} - i\varepsilon$$

$$x_{i+1} = \frac{\varepsilon}{1 \times 2}, x_{i+2} = \frac{\varepsilon}{2 \times 3}, \dots, x_n = \frac{\varepsilon}{(n-i)(n-i+1)}$$

$$\text{知 } x_1 > x_2 > \dots > x_i > x_{i+1} > \dots > x_n$$

$$x_1 + x_2 + \dots + x_n < 1 - \varepsilon + \left(\frac{1}{1 \times 2} + \frac{1}{2 \times 3} + \dots + \frac{1}{(n-i)(n-i+1)} \right) \varepsilon < 1$$

$$\text{故 } a_1 \left(\frac{1}{i} - \varepsilon \right)^3 + a_2 \left(\frac{1}{i} - 2\varepsilon \right)^3 + \dots + a_i \left(\frac{1}{i} - i\varepsilon \right)^3$$

$$+ a_{i+1} \left(\frac{\varepsilon}{1 \times 2} \right)^3 + a_{i+2} \left(\frac{\varepsilon}{2 \times 3} \right)^3 + \dots + a_n \left(\frac{\varepsilon}{(n-i)(n-i+1)} \right)^3 < 1$$

$$\text{令 } \varepsilon \rightarrow 0, \text{ 有 } a_1 + a_2 + \dots + a_i \leq i^3$$

$$\text{上式从 } 0 \text{ 加到 } n, na_1 + (n-1)a_2 + \dots + a_n \leq \frac{n^2(n+1)^2}{4}$$

$$\text{下证可以取到, 令 } a_i \leq i^3 - (i-1)^3, i=1, 2, \dots, n$$

$$\text{知 } na_1 + (n-1)a_2 + \dots + a_n = \frac{n^2(n+1)^2}{4}, \text{ 只需证}$$

$$\text{对 } x_1 > x_2 > \dots > x_n > 0, \text{ 有: } \sum_{k=1}^n (k^3 - (k-1)^3) x_k^3 \leq (x_1 + x_2 + \dots + x_n)^3$$

对 $x_1 > x_2 > \dots > x_n > 0$, 有 $\sum_{k=1}^n (k^3 - (k-1)^3) x_k^3 \leq (x_1 + x_2 + \dots + x_n)^3$
 对 n 归纳, 假设 n 成立

$$\text{即 } \sum_{k=1}^n (k^3 - (k-1)^3) x_k^3 \leq (x_1 + x_2 + \dots + x_n)^3$$

$n+1$ 时, 只需证 $((n+1)^3 - n^3) x_{n+1}^3 \leq (x_1 + x_2 + \dots + x_{n+1})^3 - (x_1 + x_2 + \dots + x_n)^3$

$$\text{右边} = x_{n+1} [(x_1 + x_2 + \dots + x_{n+1})^2 + (x_1 + x_2 + \dots + x_{n+1})(x_1 + x_2 + \dots + x_n) + (x_1 + x_2 + \dots + x_n)^2]$$

$$\geq x_{n+1} ((n+1)^2 x_{n+1}^2 + n(n+1) x_{n+1}^2 + n^2 x_{n+1}^2)$$

$$= ((n+1)^3 - n^3) x_{n+1}^3 \quad \#$$

3. 先猜答案

$$x_i = i \text{ 时, } a = \frac{1}{2}$$

$x_0 = 0$ 猜测 x_i 含 i , 又 $\frac{i+1}{x_i}$, 猜测 x_i 中含 $i+1$

$$x_i = i(i+1) \text{ 时, } a = \frac{1}{2}$$

再考虑 $x_i = i(i+1)(i+2)$

$$\text{此时有 } \frac{1}{3} (1 - \frac{1}{n+1}) \geq \frac{a}{2} (1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2})$$

令 $n \rightarrow +\infty$ 知 $a \leq \frac{4}{9}$, 猜测此为最值.

证明: 结合取等条件 $x_i - x_{i-1} : x_{i-1} = 3 : i-1$

$$\text{配成柯西: } \frac{4}{x_i - x_{i-1}} + \frac{(i-1)^2}{x_{i-1}} \geq \frac{(i+1)^2}{x_i}, \quad i=2, 3, \dots, n$$

$$\text{求和得 } 9 \sum_{i=2}^n \frac{1}{x_i - x_{i-1}} + \frac{9}{x_i} \geq 4 \sum_{i=1}^n \frac{i+1}{x_i} + \frac{n^2}{x_n} > 4 \sum_{i=1}^n \frac{i+1}{x_i}$$

$$\therefore \sum_{i=1}^n \frac{1}{x_i - x_{i-1}} \geq \frac{4}{9} \sum_{i=1}^n \frac{i+1}{x_i}$$

4. $i=1, 2, \dots, 2n-1$ 时, $|a_{i+1}-a_i|$ 互不相同

$$1 \leq |a_{i+1}-a_i| \leq 2n-1$$

故 $i=1, 2, \dots, 2n-1$ 时 $|a_{i+1}-a_i|$ 取 $1, 2, \dots, 2n-1$ 各一次.

$$\sum_{i=1}^{2n-1} |a_{i+1}-a_i| = 1+2+\dots+(2n-1) = 2n^2-n$$

充分性: 设 $1 \leq a_{2k} \leq n$, 此时 a_{2k+1} 取遍 $n+1, n+2, \dots, 2n$

$$\sum_{i=1}^{2n-1} |a_{i+1}-a_i| = \sum_{i=0}^n |a_{2i+1}-a_{2i}| + \sum_{i=1}^{n-1} |a_{2i+1}-a_{2i}|$$

$$\sum_{i=1}^{2n-1} |a_{i+1}-a_i| + |a_1-a_{2n}| = 2 \left(\sum_{i=1}^n a_{2i+1} - \sum_{i=1}^n a_{2i} \right)$$

$$= 2 \left(\sum_{i=1}^n (n+i) - \sum_{i=1}^n i \right)$$

$$= 2n^2$$

$$\text{故知 } a_1 - a_{2n} = n$$

必要性: 若 $a_1 - a_{2n} = n$, 知 $\sum_{i=1}^{2n-1} |a_{i+1}-a_i| + |a_1-a_{2n}| = 2n^2$

$$\sum_{i=1}^{2n-1} |a_{i+1}-a_i| + |a_1-a_{2n}| = 2n^2$$

注意到 $|x-y| = x+y-2\min\{x, y\}$ 代入

$$\text{知 } \sum_{i=1}^{2n-1} \min\{a_{i+1}, a_i\} + \min\{a_1, a_{2n}\} = 2(1+2+\dots+n)$$

注意到每个正整数最多在左边出现 2 次.

从而 $1, 2, \dots, n$ 均在 $\min\{a_{i+1}, a_i\}$ ($1 \leq i \leq 2n-1$) 和

$\min\{a_1, a_{2n}\}$ 中恰出现 2 次.

从而每一个不大于 n 的数两边必是大于 n 的数

每一个大于 n 的数两边也必是不大于 n 的数

注意到 $a_1 = a_{2n} + n > n$, 故 $a_2 \leq n$, $a_3 > n$, $a_{2n-1} > n$, $a_{2n} \leq n$

从而 $1 \leq a_{2k} \leq n$, 对 $k=1, 2, \dots, n$

5. 考虑 a_k 除以 $k+1$

$a_1=2$ 时, 归纳可证 $a_k=k+1$, 满足题设

$a_1<2$ 时, 归纳可证 $a_k<k+1$

令 $b_k = \frac{a_{k+1}}{k+1}$, 知 $b_k^2 > b_{k+1}$ 等价于 $a_k^2 > \frac{(k+1)^2}{k+2} a_{k+1}$

等价于 $1 > \frac{a_{k+1}}{k+2}$, 该式成立

从而 $b_k < b_{k-1}^2 < \dots < b_1^{2^{k-1}}$, 即 $a_k < (k+1)b_1^{2^{k-1}}$, 该式成立

注意到 $b_1 < 1$, 存在 k 使得 $(k+1)b_1^{2^{k-1}} < 1$, 则 $a_k < 1$

与 $a_k^2 = 1 + k a_{k+1} > 1$ 矛盾

$a_1 > 2$ 时, 归纳可证 $a_k > k+1$

知 $b_k^2 < b_{k+1}$, 等价于 $a_k^2 < \frac{(k+1)^2}{k+2} a_{k+1}$

等价于 $1 < \frac{a_{k+1}}{k+2}$, 该式成立

$b_k > b_{k-1}^2 > \dots > b_1^{2^{k-1}}$ 即 $a_k > (k+1)b_1^{2^{k-1}}$

由 $b_1 > 1$, 知存在正整数 N , $k > N$ 时, $b_1^{2^{k-1}} > e^{\frac{2}{k}}$

此时 $a_k > e^{\frac{2}{k} \cdot 2^{k-1}} = e^{\frac{2^k}{k}}$ 矛盾 #

6. $x_{k-1} \leq 1$, 对 $1 \leq k \leq n$, 又 $a_k \geq 1$

故 $\frac{1}{x_k} = 1 + a_k x_{k-1} \leq a_k + x_{k-1}$ 排序不等式

从而 $\sum_{k=1}^n \frac{1}{x_k} \leq \sum_{k=1}^n a_k + \sum_{k=1}^n x_{k-1} < \sum_{k=1}^n a_k + 1 + \sum_{k=1}^n x_k < A + \sum_{k=1}^n x_k$

由柯西不等式 $\sum_{k=1}^n \frac{1}{x_k} \cdot \sum_{k=1}^n x_k \geq n^2$

$\sum_{k=1}^n x_k (A + \sum_{k=1}^n x_k) > \sum_{k=1}^n x_k \sum_{k=1}^n \frac{1}{x_k} \geq n^2$

由二次方程求根公式:

$$\frac{n}{k} x_k > \frac{-A + \sqrt{A^2 + 4n^2}}{2} = \frac{2n^2}{A + \sqrt{A^2 + 4n^2}} > \frac{2n^2}{A + (A + \frac{2n^2}{A})} = \frac{n^2 A}{n^2 + A^2}$$

$$(\sqrt{A^2 + 4n^2} \leq \sqrt{A^2 + 2A \cdot \frac{2n^2}{A} + (\frac{2n^2}{A})^2} = A + \frac{2n^2}{A})$$

$$7. x_{k-1} - (2 - b_k) x_k + x_{k+1} = 0$$

$$x_{k-1} - 2x_k + x_{k+1} = -b_k x_k$$

$$\text{设 } a_k = -b_k x_k, \quad x_{k-1} - 2x_k + x_{k+1} = a_k$$

$$\begin{cases} -2x_1 + x_2 = a_1 \\ x_1 - 2x_2 + x_3 = a_2 \\ \dots \\ x_{n-1} - 2x_n = a_n \end{cases}$$

将 $a_1, a_2, \dots, a_n$ 看成常数, 求 x_k

$$-2x_1 + x_2 = a_1, \quad x_1 - 2x_2 + x_3 = a_2, \quad -3x_2 + 2x_3 = a_1 + 2a_2$$

$$x_2 - 2x_3 + x_4 = a_3, \quad -3x_3 + 3x_4 = a_1 + 2a_2 + 3a_3, \dots$$

$$\text{可得 } a_1 + 2a_2 + 3a_3 + \dots + k a_k = -(k+1)x_k + k x_{k+1}$$

$$a_n + 2a_{n-1} + 3a_{n-2} + \dots + (n-k) a_{k+1} = (n-k)x_k - (n-k+1)x_{k+1}$$

$$\text{消去 } x_{k+1} \text{ 得 } -(n+1)x_k = (n+1-k)(a_1 + 2a_2 + \dots + k a_k) + k((n-k)a_{k+1} + \dots + a_{n+1})$$

取 k 使 $|x_k| = \max_{1 \leq i \leq n} |x_i|$, 知 $|x_n| > 0$

$$\text{则 } (n+1)|x_k| \leq (n+1-k)|b_1||x_1| + 2(n+1-k)|b_2||x_2| + \dots + k(n+1-k)|b_k||x_k| + k(n-k)|b_{k+1}||x_{k+1}| + \dots + k|b_n||x_n|$$

$$\leq ((n+1-k)b_1 + (n+1-k)b_2 + \dots + k(n+1-k)b_k + k(n-k)b_{k+1} + \dots + kb_n) |x_k|$$

$$\begin{aligned} b_{2n+1} &\leq (n+1-k)b_1 + 2(n+1-k)b_2 + \dots + k(n+1-k)b_k + k(n-k)b_{k+1} + \dots + kb_n \\ &\leq k(n+1-k)(b_1 + b_2 + \dots + b_n) \leq \frac{(n+1)^2}{4}(b_1 + b_2 + \dots + b_n) \end{aligned}$$

$$\therefore b_1 + b_2 + \dots + b_n \geq \frac{4}{n+1} \#$$

8. 定义 $a(0)=0$

引理: 对任意非负整数 $n_1, n_2, \dots, n_k$ 有

$$1^\circ a\left(\sum_{i=1}^k n_i\right) \leq \sum_{i=1}^k 2^i a(n_i)$$

$$2^\circ a\left(\sum_{i=1}^k n_i\right) \leq 2^k \sum_{i=1}^k a(n_i)$$

证明: 1° 归纳即可

$$2^\circ \text{ 归纳不难证 } a\left(\sum_{i=1}^{2^d} n_i\right) \leq 2^d \sum_{i=1}^{2^d} a(n_i)$$

设 $d \geq 0, 2^{d-1} < k \leq 2^d$, 则

$$\begin{aligned} a\left(\sum_{i=1}^k n_i\right) &= a\left(\sum_{i=1}^k n_i + 0 + 0 + \dots + 0\right) \\ &\leq 2\left(a\left(\sum_{i=1}^{2^{d-1}} n_i\right) + a\left(\sum_{i=2^{d-1}+1}^k n_i + 0 + 0 + \dots + 0\right)\right) \\ &\leq 2\left(2^{d-1} \sum_{i=1}^{2^{d-1}} a(n_i) + 2^{d-1} \sum_{i=2^{d-1}+1}^k a(n_i) + 2^{d-1} \cdot 0 + 2^{d-1} \cdot 0 + \dots\right) \end{aligned}$$

回到原题

设 $0 = m_0 < m_1 < \dots$ 递增无界

对 $n = \sum_{i=1}^d \varepsilon_i 2^i, \varepsilon_i \in \{0, 1\}, i > d$ 时定义 $s_i = 0$

取 f 使 $M_f > d$, 知 $a(n) = a\left(\sum_{k=1}^f \sum_{M_{k-1} < i \leq M_k} \varepsilon_i 2^i\right)$

$$\leq \sum_{k=1}^f 2^k \left(\sum_{M_{k-1} < i \leq M_k} \varepsilon_i 2^i \right) \quad (\text{由引理 } 1^\circ)$$

$$\leq \sum_{k=1}^f 2^k \cdot 2(M_k - M_{k-1} + 1) \sum_{M_{k-1} < i \leq M_k} \varepsilon_i a(2^i) \quad (\text{由引理 } 2^\circ)$$

$$\left(\sum_{k=1}^f 2^k \cdot 2(M_k - M_{k-1} + 1)^2 \frac{1}{(M_{k-1} + 1)^c} \right)$$

$$\leq \sum_{k=1}^f 2^k \cdot 2(M_k + 1)^2 \frac{1}{(M_{k-1} + 1)^c}$$

$$= \sum_{k=1}^f \left(\frac{M_k + 1}{M_{k-1} + 1} \right)^2 \frac{2^{k+1}}{(M_{k-1} + 1)^{c-2}} \quad (\text{要让 } (M_{k-1} + 1)^{c-2} \text{ 压住 } 2^{k+1})$$

取 $M_k = 4^{\frac{k}{c-2}} - 1$, 上式即 $8 \times 4^{\frac{1}{c-2}} \sum_{k=1}^f \frac{1}{2^k} < 8 \times 4^{\frac{1}{c-2}}$

即知 $\{a(n)\}$ 有界 *

三元四元不等式

1. 柯: $(x^2+y+1)(1+y+z^2) \geq (x+y+z)^2$

$$\frac{1}{x^2+y+1} \leq \frac{1+y+z^2}{(x+y+z)^2}$$

$$\text{从而 } \sum \frac{1}{x^2+y+1} \leq \frac{3+\sum x+\sum x^2}{(x+y+z)^2} = \frac{\sum x^2+\sum x+3}{\sum x^2+2\sum x}$$

注意到 $\sum x = \sum xy \leq \frac{(\sum x)^2}{3}$, 知 $\sum x \geq 3$, 从而上式 ≤ 1

$x=y=z=1$ 时取等 #

2. $a=b=\frac{3}{2}$, $c=0$ 时, $ab^2+bc^2+ca^2=\frac{27}{8}$

下证 $ab^2+bc^2+ca^2 \leq \frac{27}{8}$

知 $(a^2b+b^2c+c^2a)-(ab^2+bc^2+ca^2)$

$$= (ab-ac)(a-b) + (bc-ac)(b-c)$$

$$= (a-b)(b-c)(a-c) \geq 0$$

从而 $ab^2+bc^2+ca^2 \leq \frac{1}{2}(ab^2+bc^2+ca^2+a^2b+b^2c+c^2a)$

$$= \frac{1}{2}(ab(a+b+c) + bc(a+b+c) + ca(a+b+c) - 3abc)$$

$$= \frac{3}{2}(ab+bc+ca-abc)$$

$$\leq \frac{3}{2}(ab+ca) \quad (\text{注意 } a \geq 1, bc \leq abc)$$

$$\leq \frac{3}{2} \left(\frac{a+b+c}{2} \right)^2 = \frac{27}{8} \quad \#$$

3. (1) 伯努利不等式 $a^k \leq 1+k(a-1)$

从而 $ab^k+bc^k+ca^k \leq c(1+k(a-1)) + a(1+k(b-1)) + b(1+k(c-1))$

$$= k(ab+bc+ca) + (1-k)(ab+bc+ca) = 3$$

$a=b=c$ 时取等.

(2) $b=0$ 时, ca^k 最大 $\frac{3^{k+1}k}{(k+1)^{k+1}}$

对应 $a = \frac{3k}{k+1}$, $c = \frac{3}{k+1}$

$a \geq c \geq b$ 时, 我们证明: $ab^k + bc^k + ca^k \leq c(a+b)^k$

$$c(a+b)^k \geq ca^{k-2}(a+b)^2$$

$$= ca^{k-2}(a^2 + 2ab + b^2)$$

$$\geq ca^k + a^{k-1}bc + a^{k-1}bc$$

$$\geq ca^k + bc^k + ab^k \quad (a^{k-1}bc \geq ab^k \text{ 需要 } c \geq b)$$

$a \geq b \geq c$ 时, 我们证明: $ab^k + bc^k + ca^k \leq a^k b + b^k c + c^k a$

$$\text{上式} \Leftrightarrow a^k(b-c) + c^k(a-b) \geq b^k(a-c)$$

$$\Leftrightarrow (a^k - b^k)(b-c) \geq (b^k - c^k)(a-b)$$

$$a \neq b, b \neq c \text{ 时, 上式即 } \frac{a^k - b^k}{a-b} \geq \frac{b^k - c^k}{b-c}$$

$$\text{不全证 } \frac{a^k - b^k}{a-b} \geq kb^{k-1} \geq \frac{b^k - c^k}{b-c}$$

$$\left(\text{证明: } (b+x)^k - b^k - kb^{k-1}x \text{ 记为 } f(x), x \geq 0 \right)$$

$$f'(x) = k(b+x)^{k-1} - kb^{k-1} \geq 0 \quad \#$$

化简 $a \geq c \geq b$ 情形

$$\text{而 } c(a+b)^k = \frac{1}{k} \cdot kc(a+b)^k \leq \frac{1}{k} \left(\frac{kc + k(a+b)}{k+1} \right)^{k+1} = \frac{3^{k+1}k}{(k+1)^{k+1}}$$

4. 不等式右边出现 $\sqrt{abc}$, 很难证, 换元.

$$x = \frac{1}{a}, y = \frac{1}{b}, z = \frac{1}{c}$$

$$\text{有 } \frac{\sqrt{a^3+b^3}}{ab+1} = \frac{\sqrt{\frac{x^3+y^3}{xy}}}{xy+1} \geq \frac{\sqrt{\frac{x^2y+xy^2}{xy}}}{xy+1} = \frac{\sqrt{xy}}{xy+1}$$

$$\text{只需证 } xy + yz + zx = 1 \text{ 时, } \sum \frac{x}{x^2+1} \leq \frac{1}{3\sqrt{2xyz}} \leq \frac{\sqrt{x+y}}{xy+1}$$

$$\text{知 } \frac{x}{x^2+1} = \frac{x}{x^2+xy+yz+zx} = \frac{x}{(x+y)(x+z)}$$

发现 x^2+1, y^2+1 相对好处理

柯西不等式化为 $xy+1$

$$xy+1 \leq \sqrt{(x^2+1)(y^2+1)} = \sqrt{(x+y)^2(y+z)(z+x)}$$

$$\text{故 } \frac{1}{3\sqrt{2xyz}} \geq \frac{\sqrt{xy}}{xy+1}$$

$$\geq \frac{1}{3\sqrt{2xyz}} \geq \frac{\sqrt{xy}}{\sqrt{(x+y)(y+z)(z+x)}}$$

$$= \frac{1}{\sqrt{2xyz(x+y)(y+z)(z+x)}}$$

$$\geq \frac{1}{(x+y)(y+z)(z+x)} \quad \#$$

$$5. \text{知 } \frac{c}{a+2b} + 1 = \frac{a+2b+c}{a+2b}$$

$$\frac{a+2b}{c} + 1 = \frac{a+2b+c}{c}, \text{统一分子后求和}$$

$$\text{故 } \sum \frac{c}{a+2b} + \sum \frac{a+2b}{c} = \sum \left(\frac{a+2b+c}{a+2b} + \frac{a+2b+c}{c} \right) - 8$$

$$= \sum \frac{(a+2b+c)^2}{c(a+2b)} - 8$$

$$\text{柯西不等式经典形式 } \sum_{i=1}^n \frac{a_i^2}{b_i} \geq \frac{(\sum a_i)^2}{\sum b_i}$$

$$\text{上式} \geq \frac{(\sum (a+2b+c))^2}{\sum c(a+2b)} - 8$$

$$= 8 \left(\frac{(a+b+c+d)^2}{a^2+b^2+c^2+d^2+ab+ac+ad+bc+bd+cd} - 1 \right) \quad \#$$

6. $a=b=c=d=1$ 时取到 $\frac{4}{3}$

$c=d=0$ 时 $\frac{4-b}{b^2+4} + \frac{b}{4}$ 不难看出 $a=b=2$ 时最大 $\frac{2}{3}$.

下证最小值为 $\frac{2}{3}$ 即 $\frac{a}{b^3+4} + \frac{b}{c^3+4} + \frac{c}{d^3+4} + \frac{d}{a^3+4} \geq \frac{2}{3}$

由均值不等式 $b^3+4 = \frac{b^3}{2} + \frac{b^3}{2} + 4 \geq 3b^2$

$$\text{故 } \frac{4a}{b^3+4} \leq a - \frac{ab^3}{b^3+4} \geq a - \frac{ab}{3}$$

$$\sum \frac{a}{b^3+4} \geq \frac{\sum a}{4} - \frac{\sum ab}{12}$$

$$\text{而 } \sum ab = (a+c)(b+d) \leq \left(\frac{a+b+c+d}{2}\right)^2 = 4$$

$$\text{故 } \sum \frac{a}{b^3+4} \geq 1 - \frac{4}{12} = \frac{2}{3}, \text{ 即知 } \frac{2}{3} \text{ 为最小值}$$

7. 不妨设 $a_1 \leq a_2 \leq a_3 \leq a_4$

$$\text{知 } a_1^2 + a_2^2 + a_3^2 \geq 2(a_1a_2 + a_2a_3 + a_3a_1)$$

$$\Leftrightarrow (a_3 + a_2)^2 + a_1^2 \geq 2a_1(a_2 + a_3) + 4a_2a_3$$

$$\Leftrightarrow (a_3 + a_2 - a_1)^2 \geq (2\sqrt{a_2a_3})^2$$

$$\Leftrightarrow a_3 + a_2 - a_1 \geq 2\sqrt{a_2a_3}$$

$$\Leftrightarrow \sqrt{a_3} \geq \sqrt{a_1} + \sqrt{a_2}$$

类似地, 有 $\sqrt{a_4} \geq \sqrt{a_3} + \sqrt{a_2}$

设 $a_2 = \beta^2, a_3 = \gamma^2, 1 \geq \beta > 0$ 有:

$$a_1 \leq \min\{(\gamma - \beta)^2, \beta^2\}, a_4 = (\beta + \gamma)^2 \text{ 时 } \frac{\sum_{i=1}^4 a_i^2}{\sum_{1 \leq i < j \leq 4} a_i a_j} \text{ 取最大值}$$

允许 $a_1 = 0$

$$\text{若 } \gamma \leq 2\beta \text{ 有 } \sum_{i=1}^4 a_i^2 = 3(\beta^4 + 4\beta^2\gamma^2 + \gamma^4)$$

$$\sum_{1 \leq i < j \leq 4} a_i a_j = 3(\beta^4 + \beta^2\gamma^2 + \gamma^4)$$

$$\frac{\beta^4 + 4\beta^2\gamma^2 + \gamma^4}{\beta^4 + \beta^2\gamma^2 + \gamma^4} = 1 + \frac{3}{\frac{\beta^2}{\gamma^2} + \frac{\gamma^2}{\beta^2} + 1} \geq 1 + \frac{3}{\varphi + \frac{1}{\varphi} + 1} = \frac{11}{7}$$

$a_1=1, a_2=1, a_3=4, a_4=9$ 时可以取到 $\frac{11}{7}$

若 $r > 2\beta$, $a_1 = \beta^2$

$$\sum_{i=1}^4 a_i^2 = 2r^4 + 4\beta r^3 + 6\beta^2 r^2 + 4\beta^3 r + 3\beta^4$$

$$\sum_{1 \leq i < j \leq 4} a_i a_j = r^4 + 2\beta r^3 + 5\beta^2 r^2 + 4\beta^3 r + 3\beta^4$$

$$\sum_{i=1}^4 a_i^2 \geq \frac{11}{7} \sum_{1 \leq i < j \leq 4} a_i a_j \Leftrightarrow 3r^4 + 6\beta r^3 - 13\beta^2 r^2 - 16\beta^3 r + 14\beta^4 \geq 0$$

由 $r > 2\beta$, 上式成立

下证: $a_1 \leq \min\{(r-\beta)^2, \beta^2\}$ 且 $a_4 \geq 2(\beta+r)^2$ 条件下, 仍有

$$\sum_{i=1}^4 a_i^2 \geq \frac{11}{7} \sum_{1 \leq i < j \leq 4} a_i a_j, \text{ 令 } F = \sum_{i=1}^4 a_i^2 - \frac{11}{7} \sum_{1 \leq i < j \leq 4} a_i a_j$$

固定 a_2, a_3, a_4 , 知 $a_1 < \frac{1}{4}(a_2+a_3+a_4)$ 时, F 关于 a_1 递减

$$\text{而 } \min\{(r-\beta)^2, \beta^2\} \leq \beta^2 = a_2 < \frac{1}{4}(a_2+a_3+a_4)$$

故只要证 $a_1 = \min\{(r-\beta)^2, \beta^2\}$ 时 $F \geq 0$

固定 a_1, a_2, a_3 当 $a_4 \geq \frac{1}{4}(a_1+a_2+a_3)$ 时, F 关于 a_4 递增

$$\frac{1}{4}(a_1+a_2+a_3) \leq \frac{1}{4}(\beta^2+\beta^2+r^2) \leq \frac{1}{4}(\beta+r)^2 < (\beta+r)^2$$

只要 $a_4 = (\beta+r)^2$, $F \geq 0$, #

8. 设 $x = \sqrt{b} + \sqrt{c} - \sqrt{a}$, $y = \sqrt{c} + \sqrt{a} - \sqrt{b}$, $z = \sqrt{a} + \sqrt{b} - \sqrt{c}$

显然 $x > 0, y > 0, z > 0$, 且 $a = (\frac{y+z}{2})^2$, b, c 类似

$$\text{从而 } b+c-a = (\frac{z+x}{2})^2 + (\frac{x+y}{2})^2 - (\frac{y+z}{2})^2 = \frac{x^2+xy+xz-yz}{2}$$

$$\frac{b+c-a}{b+c-a} = \frac{\sqrt{x^2+xy+xz-yz}}{\sqrt{2x^2}} = \sqrt{1 - \frac{(x-y)(x-z)}{2x^2}} \leq 1 - \frac{(x-y)(x-z)}{4x^2}$$

(经典去根号方式: $\sqrt{1-x} \leq 1 - \frac{x}{2}$) 另外两式类似

三式相加, 只要证 $\sum \frac{(x-y)(x-z)}{x^2} \geq 0$ (直接相消也可化到此步)

不妨设 $x \geq y \geq z$

$$\text{知 } \frac{(z-x)(z-y)}{z^2} \geq \frac{(x-z)(y-z)}{y^2} \geq \frac{(x-y)(y-z)}{y^2} = -\frac{(y-z)(y-x)}{y^2}$$

$$\frac{(x+y)(x-z)}{x^2} \geq 0 \text{ 或求和 } \#$$

代数杂题

1. 设 $t = \lambda^{\frac{1}{3}}$, $t^3 = t^2 + 1$

$f(x) = x^3 - x^2 - 1$ 只有唯一实根 t , $f(\sqrt{2}) < 0$, 故 $t > \sqrt{2}$

设 u, v 为 $f(x) = 0$ 另外两复根, u, v 为共轭复数

$\therefore tuv = 1$, 故 $|u| = |v| = |t|^{-\frac{1}{2}} < 2^{-\frac{1}{4}}$

取 $M = u^{900} + v^{900} + t^{900}$

由 u, v, t 是一整系数多项式 $f(x)$ 的三个根, 知 M 是整数

简单证明知 $u^{n+3} = u^{n+2} + u^n$, 其它两式类似

三式求和: 设 $a_j = u^j + v^j + t^j$, 递推 $a_{j+3} = a_{j+2} + a_j$

不难算出 a_0, a_1, a_2 均为整数, 归纳即知 a_j 均为整数

$|M - \lambda^{300}| = |M - t^{900}| = |u^{900} + v^{900}| < 2 \cdot 2^{-\frac{900}{4}} < \frac{1}{4100}$

2. 若 $x = 0$, 由前式知 $y = 0$, 不满足后式, 故 $x \neq 0$, 同理 $y \neq 0$

由 $x(1-2y) \geq 0$, $y(1-2x) \geq 0$

若 $x < 0$, 由 $1-2x > 0$ 知 $y > 0$

此时 $\frac{2}{\sqrt{1+2xy}} > 2 > \frac{1}{\sqrt{1+2x^2}} + \frac{1}{\sqrt{1+2y^2}}$, 矛盾

故 $x > 0$, 同理 $y > 0$, 知 $0 < x, y \leq \frac{1}{2}$

由 $\frac{1}{\sqrt{1+2x^2}} + \frac{1}{\sqrt{1+2y^2}} = \frac{2}{\sqrt{1+2xy}}$

知 $\frac{1}{1+2x^2} + \frac{1}{1+2y^2} \geq \frac{2}{1+2xy}$ 柯西

$(\frac{1}{1+2x^2} - \frac{1}{1+2xy}) + (\frac{1}{1+2y^2} - \frac{1}{1+2xy}) \geq 0$

$\frac{x(y-x)}{(1+2xy)(1+2x^2)} + \frac{y(x-y)}{(1+2xy)(1+2y^2)} \geq 0$

$$\text{即 } \frac{(x-y)^2(1-2xy)}{(1+2xy)(1+2x^2)(1+2y^2)} \leq 0$$

$$\text{由 } 1-2xy > 1 - \frac{1}{2} \cdot \frac{1}{2} > 0 \text{ 知 } (x-y)^2 \leq 0, x=y$$

代入后一式, 知 $x=y=\frac{4}{9}$ 或 $\frac{1}{18}$, 经检验, 均符合题设 #

$$3(4) \bar{a}_1 = 00001 \ 00002 \ 00003 \ \dots \ 10000$$

$$a_2 = 00002 \ 00003 \ 00004 \ \dots \ 10001$$

!

$$a_{2022} = 02022 \ 02023 \ 02024 \ \dots \ 12021$$

$$* \text{法} = (\text{我记的}) \quad \bar{a}_1 = 8888888811111111$$

$$\{a_n\} \text{公差 } d = 100000009$$

(2) 不存在

若 $a_1, a_2, \dots$ 满足要求

$$\text{设 } d \text{ 为公差 } a_{n+1} = a_1 + nd$$

$$\text{设 } a \text{ 是 } k \text{ 位数, 取 } n = 10^{k+1}$$

知 $a + 10^{k+1}d$ 数字和为 a 的数字和为 a 的数字和加上 d 的数字和

$$\text{再取 } n = 10^{k+2}$$

$$a + 10^{k+2}d \text{ 数字和也为 } a \text{ 的数字和加上 } d \text{ 的数字和}$$

$$10^{k+2} > 10^{k+1} \text{ 矛盾.}$$

4. 注意 $\sqrt{a_{i+1}^2 + a_{i+1}a_{i+2} + a_{i+2}^2 + a_j a_{i+1}}$ 中的 $a_i a_{i+1}$ 换成 $a_i a_{i+2}$

这一项就会变成 $a_{i+2} + a_{i+2}$ 很好处理

$\sqrt{a_i^2 + a_i a_{i+1} + a_{i+1}^2} + \sqrt{a_{i+1}^2 + a_{i+1} a_{i+2} + a_{i+2}^2}$ 中的 $a_{i+1} a_{i+2}$ 则希望换成 $a_i a_{i+1}$ 两项搭配, 发现引理中的交换条件即可.

引理: 非负实数 x, y, z, w 满足 $(x-y)(z-w) \geq 0$ 时

$$\sqrt{x+w} + \sqrt{y+z} \geq \sqrt{x+z} + \sqrt{y+w} \quad (\text{平方即知})$$

回到原题, 对 $\forall i \in \mathbb{Z}$.

$$\begin{aligned} & ((a_{i+1}^2 + a_{i+1} a_{i+2} + a_{i+2}^2) - (a_i^2 + a_i a_{i+1} + a_{i+1}^2)) (a_{i+1} a_{i+2} - a_i a_{i+1}) \\ &= a_{i+1} (a_{i+2} - a_{i+1})^2 (a_i + a_{i+1} + a_{i+2}) \geq 0 \end{aligned}$$

$$\begin{aligned} & \text{由引理 } \sqrt{a_{i+1}^2 + a_{i+1} a_{i+2} + a_{i+2}^2} + \sqrt{a_i^2 + a_i a_{i+1} + a_{i+1}^2} \\ & \geq \sqrt{a_{i+1}^2 + a_{i+1} a_{i+2} + a_{i+2}^2} + \sqrt{a_i^2 + a_i a_{i+1} + a_{i+1}^2} \\ & = a_i + 2a_{i+1} + a_{i+2} \end{aligned}$$

$$\begin{aligned} & \text{故 } \max \left\{ \sum_{i=1}^4 \sqrt{a_i^2 + a_i a_{i+1} + a_{i+1}^2}, \sum_{i=1}^4 \sqrt{a_i^2 + a_i a_{i+1} + a_{i+1}^2} \right\} \\ & \geq \frac{1}{2} \sum_{i=1}^4 (a_i + 2a_{i+1} + a_{i+2}) \\ & = \frac{1}{2} \sum_{i=1}^4 (a_i + 2a_{i+1} + a_{i+2}) \\ & = 2 \end{aligned}$$

5. 猜测 $a_n^2 = (b_{n+1})^3 - b_n^3$

1) 归纳若对 n 成立, 则对 $n+1$: $(b_{n+1} + 1)^3 - b_{n+1}^3 =$

$$\begin{aligned} & (b_{n+1} + 1)^3 - b_{n+1}^3 = 48a_n^2 + 147b_n^2 + (68a_n b_n + 8a_{n+1} + 47b_n + 3) \\ & = (7a_n + 12b_n + 6)^2 + (36b_n^2 + 3b_n + 1) - a_n^2 = a_{n+1}^2 \end{aligned}$$

6. 最大值 $\frac{1}{3}$. $ab^2c + a^2bc + ab^2c \leq b^2 \frac{a^2+c^2}{2} + a^2 \frac{b^2+c^2}{2} + c^2 \frac{a^2+b^2}{2}$
 $= a^2b^2 + b^2c^2 + c^2a^2 \leq \frac{1}{3}(a^2+b^2+c^2)^2 = \frac{1}{3}$
 (取等条件: $a=b=c=\frac{\sqrt{3}}{3}$)

求 $abc(a+b+c)$ 最小值

只要考虑 a, b, c 两正一负情形, 不妨设 $a > b > 0, c < 0$

只要考虑 $a+b+c > 0$

设 $x = -c$ 知 $a^2+b^2 = 1-x^2, a+b-x > 0$ 此时 $x^2 < \frac{2}{3}$

$$abx(a+b-x) \leq \frac{a^2+b^2}{2} x (\sqrt{2(a^2+b^2)} - x) = \frac{x(1-x^2)}{2} (\sqrt{2(1-x^2)} - x)$$

我们求 $f(x) = \frac{(1-x)\sqrt{x}}{2} (\sqrt{2(1-x)} - \sqrt{x})$ 在 $x \in [0, \frac{2}{3}]$ 上的最大值
 上式即 $f(x)$

$$f'(x) = \dots = \frac{1}{2\sqrt{2x}} ((1-4x)\sqrt{1-x} + (2x-1)\sqrt{2x})$$

$$\text{令 } (1-4x)\sqrt{1-x} = (1-2x)\sqrt{2x}$$

$$\text{平方得 } \dots \Rightarrow (3x-1)(8x^2-8x+1)=0$$

方程根中 $\frac{1}{3}$ 是增根, $\frac{1}{2} + \frac{\sqrt{2}}{4} > \frac{2}{3}$

故 $x = \frac{1}{2} - \frac{\sqrt{2}}{4}$ 是 $f(x)$ 最大值点

$$\text{原问题最小值} = f\left(\frac{1}{2} - \frac{\sqrt{2}}{4}\right) = -\frac{1+\sqrt{2}}{16}$$

7. 设 $s_1 = \sin \frac{2}{7}\pi, s_2 = \sin \frac{4}{7}\pi, s_3 = \sin \frac{6}{7}\pi$

(1) $s_1 + s_2 + s_3 = \frac{\sqrt{7}}{2}$

(2) $s_1 s_2 + s_2 s_3 + s_3 s_1 = 0$

(3) $s_1 s_2 s_3 = -\frac{\sqrt{7}}{8}$

先证(2): $\sin \frac{2\pi}{7} \sin \frac{4\pi}{7} + \sin \frac{4\pi}{7} \sin \frac{8\pi}{7} + \sin \frac{8\pi}{7} \sin \frac{2\pi}{7}$
 $= \frac{1}{2} (\sin \frac{2\pi}{7} - \cos \frac{6\pi}{7} + \cos \frac{12\pi}{7} + \cos \frac{6\pi}{7} - \cos \frac{10\pi}{7}) = 0$

再证(1) 知 $\sin^2 \frac{2\pi}{7} + \sin^2 \frac{4\pi}{7} + \sin^2 \frac{8\pi}{7} = \frac{1}{2} \frac{6}{7} \sin^2 \frac{2\pi}{7}$
 $= \frac{1}{4} \frac{6}{7} (1 - \cos \frac{4\pi}{7})$
 $= \frac{1}{4} (7 - \frac{7}{7} \cos \frac{4\pi}{7})$
 $= \frac{7}{4}$ 又(2)知 $(S_1 + S_2 + S_3)^2 = S_1^2 + S_2^2 + S_3^2$

又知 $S_1 + S_2 + S_3 > 0$, 即证 $S_1 + S_2 + S_3 = \frac{\sqrt{7}}{2}$

证(3): 因为 $4(\sin^3 \frac{2\pi}{7} + \sin^3 \frac{4\pi}{7} + \sin^3 \frac{8\pi}{7})$
 $= 3(\sin \frac{2\pi}{7} + \sin \frac{4\pi}{7} + \sin \frac{8\pi}{7}) - (\sin(3 \times \frac{2\pi}{7}) + \sin(3 \times \frac{4\pi}{7}) + \sin(3 \times \frac{8\pi}{7}))$
 $= 3(S_1 + S_2 + S_3) - (\sin \frac{6\pi}{7} + \sin \frac{12\pi}{7} + \sin \frac{24\pi}{7})$
 $= 3(S_1 + S_2 + S_3) + (\sin \frac{8\pi}{7} + \sin \frac{2\pi}{7} + \sin \frac{4\pi}{7})$
 $= 4(S_1 + S_2 + S_3)$

故 $S_1^3 + S_2^3 + S_3^3 = S_1 + S_2 + S_3 = \frac{\sqrt{7}}{2}$

又 $S_1^3 + S_2^3 + S_3^3 - 3S_1S_2S_3 = (S_1 + S_2 + S_3)(S_1^2 + S_2^2 + S_3^2 - S_1S_2 - S_2S_3 - S_3S_1)$

设 $x_i = \frac{\cos \frac{2i\pi}{7}}{\sqrt{7}} = \frac{1}{\sqrt{7}S_i}$, $i=1, 2, 3$

知 $x_1 + x_2 + x_3 = \frac{S_2 + S_3 + S_1}{\sqrt{7}S_1S_2S_3} = 0 \in \mathbb{Q}$

$x_1x_2 + x_2x_3 + x_3x_1 = -\frac{4}{7} \in \mathbb{Q}$

$x_1x_2x_3 = -\frac{8}{49} \in \mathbb{Q}$

由韦达定理, x_1, x_2, x_3 是 $x^3 - \frac{4}{7}x + \frac{8}{49} = 0$ 的三根

故 $x_i^n = \frac{4}{7}x_i^{n-2} - \frac{8}{49}x_i^{n-3}$, $i=1, 2, 3$

求和 $x_1^n + x_2^n + x_3^n = \frac{4}{7}(x_1^{n-2} + x_2^{n-2} + x_3^{n-2}) - \frac{8}{49}(x_1^{n-3} + x_2^{n-3} + x_3^{n-3})$
 $n=3, 4, \dots$, 归纳不难证, 对任意 $n \in \mathbb{N}^*$,

$$x_1^n + x_2^n + x_3^n \in \mathbb{Q}$$

原式即 $7^{1033} (x_1 x_2 x_3)^{20} \underbrace{(x_1^3 + x_2^3 + x_3^3)(x_1^{2003} + x_2^{2003} + x_3^{2003})}_{= (x_1^{2006} + x_2^{2006} + x_3^{2006})}, \#$

8. $f(x)$ 和 $f(y)$ 很难算, 找两者间关系

$$x \geq f(y)$$

$$\text{当 } y \geq x \text{ 时, } f(x) = \left(\sum_{a_i \leq x} c_i a_i^2 \right)^{\frac{1}{2}} + \left(\sum_{a_i > x} c_i a_i^\alpha \right)^{\frac{1}{\alpha}}$$

$$\leq \left(\sum_{a_i \leq y} c_i a_i^2 \right)^{\frac{1}{2}} + \left(\sum_{x < a_i \leq y} c_i a_i^\alpha + \sum_{a_i > y} c_i a_i^\alpha \right)^{\frac{1}{\alpha}}$$

引理: 对 $x, y \geq 0, r > 0$, 有 $(x+y)^r \leq 2^r (x^r + y^r)$

$$\text{证明: } (x+y)^r \leq (2 \max\{x, y\})^r = 2^r \max\{x^r, y^r\} \leq 2^r (x^r + y^r)$$

$$\text{从而 } f(x) \leq \left(\sum_{a_i \leq y} c_i a_i^2 \right)^{\frac{1}{2}} + 2^{\frac{1}{\alpha}} \left(\sum_{x < a_i \leq y} c_i a_i^\alpha \right)^{\frac{1}{\alpha}} + 2^{\frac{1}{\alpha}} \left(\sum_{a_i > y} c_i a_i^\alpha \right)^{\frac{1}{\alpha}}$$

$$\leq 2^{\frac{1}{\alpha}} f(y) + 2^{\frac{1}{\alpha}} \left(\sum_{x < a_i \leq y} c_i a_i^\alpha \right)^{\frac{1}{\alpha}} \quad ①$$

下面放缩 $\left(\sum_{x < a_i \leq y} c_i a_i^\alpha \right)^{\frac{1}{\alpha}}$

$$\sum_{x < a_i \leq y} c_i a_i^\alpha = \sum_{x < a_i \leq y} c_i a_i^2 \cdot a_i^{\alpha-2} \leq x^{\alpha-2} \sum_{x < a_i \leq y} c_i a_i^2$$

$$\leq x^{\alpha-2} \sum_{a_i \leq y} c_i a_i^2$$

$$\leq x^{\alpha-2} f(y)^2 \leq x^\alpha$$

利用 $0 < \alpha < 2$ 及 $f(y) \leq x$

$$\text{由 } ① \text{ 和 } f(x) \leq 2^{\frac{1}{\alpha}} f(y) + 2^{\frac{1}{\alpha}} x \leq 2^{\frac{1}{\alpha}} x$$

当 $y < x$ 时, 类似可知

$$f(x) = \left(\sum_{a_i \leq y} C_i a_i^2 + \sum_{y < a_i \leq x} C_i a_i^2 \right)^{\frac{1}{2}} + \left(\sum_{a_i > x} C_i a_i^\alpha \right)^{\frac{1}{2}}$$

$$\leq 2^{\frac{1}{2}} \left(\sum_{a_i \leq y} C_i a_i^2 \right)^{\frac{1}{2}} + 2^{\frac{1}{2}} \left(\sum_{y < a_i \leq x} C_i a_i^2 \right)^{\frac{1}{2}} + \left(\sum_{a_i > x} C_i a_i^\alpha \right)^{\frac{1}{2}}$$

$$\leq 2^{\frac{1}{2}} f(y) + 2^{\frac{1}{2}} \left(\sum_{y < a_i \leq x} C_i a_i^2 \right)^{\frac{1}{2}}$$

$$\leq 2^{\frac{1}{2}} f(y) + 2^{\frac{1}{2}} \left(x^{2-\alpha} \sum_{y < a_i \leq x} C_i a_i^\alpha \right)^{\frac{1}{2}} \quad (\text{因为 } 0 < \alpha < 2)$$

$$\leq 2^{\frac{1}{2}} f(y) + 2^{\frac{1}{2}} (x^{2-\alpha} f^\alpha(y))^{\frac{1}{2}}$$

$$\leq 2^{\frac{1}{2}} \cdot 2x$$

$$\leq 2^{\frac{\alpha+1}{2}} x \quad \#$$

数论中的代数方法

1. 已知 $d(n) \leq 2\sqrt{n}$

取 M 充分大, $M > 1$, 使 $M > 2\sqrt{M} + C$

知 $a_n < M$, 则 $a_{n+1} < 2\sqrt{a_n} + C < 2\sqrt{M} + C < M$

又 $a_1 = 1$ 知 $\{a_n\}$ 中各项均小于 M

$a_1, a_2, \dots, a_m$ 不同取值数 $\leq M-1$, 抽屉原理

必有两个相同, 设为 $a_i, a_j, i < j$

由 $a_i = a_j$ 知 $a_{i+1} = d(a_i) + C = d(a_j) + C = a_{j+1}$

$a_{i+2} = a_{j+2}, \dots$, 对 $\forall t \in \mathbb{N}^*$, $a_{i+t} = a_{j+t}$

故数列以 $j-i$ 为周期 #

2. 注意到 $a_{n+1} \geq a_n$ (注意 $f(n) > 0$)

$$a_{n+1} \geq a_n + \sqrt{a_n^2 + f(n)} > 2a_n$$

$$\text{有 } (a_{n+1} - a_n)^2 = a_n a_{n+1} + f(n)$$

$$f(n) = a_{n+1}^2 - 3a_n a_{n+1} + a_n^2$$

$$f(n+1) = a_{n+2}^2 - 3a_{n+1} a_{n+2} + a_{n+1}^2$$

$$\text{知 } |a_{n+2} - a_n| \geq a_{n+2} - a_n \geq a_{n+1} \geq 2^n a_1 \geq 2^n$$

$$\text{存在正整数 } N, n > N \text{ 时 } 2^n > |f(n) - f(n+1)|$$

$$\text{故 } a_{n+2} + a_n - 3a_{n+1} = 0$$

$$f(n+1) = f(n)$$

$$f(N+1) = f(N+2) = \dots$$

$f(n) = f(N+1)$ 无穷多值, 知 f 为常数

3. (注意到 $a_{n+1} = a_n^2 + a_n + c^3 \equiv a_n \pmod{a_n^2 + c^3}$)
 $a_{n+1}^2 + c^3 \equiv a_n^2 + c^3 \pmod{a_n^2 + c^3}$)

$$\begin{aligned} a_{n+1}^2 + c^3 &= (a_n^2 + a_n + c^3)^2 + c^3 \\ &= (a_n^2 + c^3)(a_n^2 + c^3 + 2a_n) + a_n^2 + c^3 \\ &= (a_n^2 + c^3)(a_n^2 + 2a_n + 1 + c^3) \end{aligned}$$

下证: $(a_n^2 + c^3, a_n^2 + 2a_n + 1 + c^3) = 1$

反证, 设 $p \mid a_n^2 + c^3, p \mid a_n^2 + 2a_n + 1 + c^3$

知 $p \mid 2a_n + 1$

又 $p \mid 4a_n^2 + 4c^3$, 知 $p \mid 4c^3 + 1$

$$\begin{aligned} p \mid a_n + 2 &= 4(a_{n-1}^2 + a_{n-1} + c^3) + 2 \\ &= (2a_{n-1} + 1)^2 + (4c^3 + 1) \end{aligned}$$

故 $p \mid (2a_{n-1} + 1)^2$, 即 $p \mid 2a_{n-1} + 1$

同理, 从 $p \mid 2a_{n-1} + 1$ 可得 $p \mid 2a_{n-2} + 1, \dots, p \mid 2a_1 + 1$

故 $(a_n^2 + c^3, a_n^2 + 2a_n + 1 + c^3) = 1$

若 $a_k^2 + c^3$ 为正整数的 M 次幂, 知

$a_{k-1}^2 + c^3$ 也是, $\dots, a_1^2 + c^3$ 也为正整数 M 次幂

$$c^2(c+1) = t^m$$

由 $(c^2, c+1) = 1$, 知 $c^2 = t_1^m, c+1 = t_2^m, t_1, t_2 \in \mathbb{N}^+$

若 M 为奇, $c = t_3^M, t_2^M - t_1^M = 1$, 与 $M > 1$ 矛盾

若 M 为偶, $c = t_2^{2M_0} - 1 = r^2 - 1$

其中设 $M = 2M_0, r = t_2^{M_0}$ 知 $r \geq 2$

而 $c = r^2 - 1$, $r \geq 2$, $r \in \mathbb{N}^*$ 时 $a_1^2 + c^3 = (r^3 - 1)^2$ 是正整数 2 次幂

所有 c 为 $r^2 - 1$, $r \geq 2$, $r \in \mathbb{N}^*$.

4. (粗略)

设 $b_n = S_{n+1} - (n+1)a_n$ 知 $b_n \in \{-1, 1\}$

$n \geq 2$ 时, $S_n = na_{n-1} + b_{n-1}$

上式 $n, n+1$ 求和, 得 $a_{n+1} = (n+1)a_n + na_{n-1} + S_n$ ①

$S_n = b_n + b_{n-1} \in \{-2, 0, 2\}$

不难证 $a_1 \geq 675$, $a_2 \geq 225$, 从而 $n \geq 0$ 时 $a_n \geq 225$

(由①式, a_{n+1} 比 $(n+1)a_n$ 大一些, 比 $(n+2)a_n$ 小一点点)
直觉 $\frac{a_{n+2}}{a_n}$ 可估到一个很小的范围

引理 1: $n \geq 4$ 时, $a_{n+2} = (n+1)(n+4)a_n$

证明: $a_n < (n+1)a_{n-1}$

由①, $n \geq 4$ 时

$$a_{n+2} = (n+2)a_{n+1} + (n+1)a_n + S_{n+1}$$

$$= (n+2)((n+1)a_n + na_{n-1} + S_n) + (n+1)a_n + S_{n+1}$$

$$> (n+1)(n+3)a_n + na_n$$

可估 $a_{n+2} < (n+1)(n+3)a_n + (n+2)a_n$, 即有

$$a_{n+2} = (n+1)(n+3)a_n + (n+2)a_n$$

即有 $a_{n+2} = (n+1)(n+3)a_n + (n+1)a_n = (n+1)(n+4)a_n$

引理2: $n \geq 4$ 时, $a_{n+1} = \frac{(n+1)(n+3)}{n+2} a_n$

引理3: $n \geq 1$ 时, $a_{n+1} = \frac{(n+1)(n+3)}{n+2} a_n$

回到原题, 令 $n=1$, $a_2 = \frac{8}{3} a_1 \in \mathbb{N}^*$

设 $a_1 = 3c$, 引理3, 求证 $a_n = (n!)(n+2)c$

又 $|s_2 - 2a_1| = 1$, 故 $a_0 = c-1$ 或 $c+1$

$\{a_n\}$ 为 $a_0 = c-1$, $n \geq 1$ 时 $a_n = (n!)(n+1)c$, 其中 $c \geq 2025$

或 $a_0 = c+1$, $n \geq 2$ 时 $a_n = (n!)(n+1)c$, 其中 $c \geq 2023$

5. 假设四个整根 $a, b, c, -(a+b+c)$

韦: $\begin{cases} 2p = ab+bc+ca - (a+b+c)(a+b+c) \\ -4 = abc - (a+b+c)(ab+bc+ca) \\ p^2 - 3b = -abc(a+b+c) \end{cases}$

$p^2 - 3b = -abc(a+b+c)$

从而 $-4p = (a+b)^2 + (b+c)^2 + (c+a)^2$

由此式 $a+b, b+c, c+a$ 均偶

设 $a+b=2u, c+a=2v, a+b=2w, u, v, w \in \mathbb{Z}$

知 $-p = u^2 + v^2 + w^2$

$\begin{cases} a = -u+v+w \\ b = u-v+w \\ c = u+v-w \end{cases}$

$p^2 - 3b = -(-u+v+w)(u-v+w)(u+v-w)(u+v+w)$
 $= u^4 + v^4 + w^4 - 2u^2v^2 - 2v^2w^2 - 2w^2u^2$

$$\bar{0} - p = u^2 + v^2 + w^2$$

$$p^2 = u^4 + v^4 + w^4 + 2u^2v^2 + 2v^2w^2 + 2w^2u^2$$

$$\text{从而 } u^2v^2 + v^2w^2 + w^2u^2 = q$$

$$(9, 0, 0) (4, 4, 1)$$

$$\text{不难求出 } p = -10, q = 0$$

$$x^4 - 20x^2 + 64 = (x+4)(x-4)(x+2)(x-2)$$

$$\text{四个整根 } p = -6, q = \pm 16$$

$$x^4 - 12x^2 + 16x = x(x-2)^2(x+4)$$

$$x^4 - 12x^2 - 16x = x(x+2)^2(x-4)$$

均有四个整根

$$\text{满足条件 } (p, q) = (-10, 0), (-6, 16), (-6, -16)$$

$$6. a|b^2, a+1|b^2+1$$

$$\text{知 } a(a+1)|b^2-a$$

$$\text{设 } b^2-a = da(a+1), d \text{ 非负整数}$$

$$b^2|a^4 \text{ 是 } d(a+1)+1|a^3$$

$$a^3 = M(d(a+1)+1)$$

$$\text{模 } a+1: M \equiv -1 \pmod{a+1}, M = e(a+1)-1, e \in \mathbb{Z}$$

$$a^2-a+1 = de(a+1) + (e-d)$$

$$\text{模 } a+1: \text{ 设 } e-d = k(a+1)+3$$

$$\text{知 } de = a-2-k$$

1° 若 $k \neq 0, -1$, 验证 $a-2-k \leq |k(a+1)+3|$

即 $de \leq |e-d|$

若 $d > 0$, $|e-d| \leq \max\{d, e\}$ 矛盾

若 $d=0$, $a=b^2$, $(a, b) = (t^2, t)$

2° 若 $k=0$, 则 $a=d^2+3d+2$, $b^2=(d+1)^4(d+2)$

故 $d+2$ 为完全平方数 $d=t^2-2$, $t \geq 2$

得 $(a, b) = (t^2(t^2-1), t(t^2-1)^2)$

经检验, 均满足

综上, $(a, b) = (t^2, t)$ 或 (t^2, t^3) ($t \geq 2$) 或 $(t^2(t^2-1), t(t^2-1)^2)$ ($t \geq 2$)

7. 设 $c(c^2-c+1) = pab$ ①

$a+b = q(c^2+1)$ ② $p, q \in \mathbb{N}^*$

不妨设 $a \leq b$

(要证 $a=c$, $b=c^2-c+1$. 如果 $b \neq c^2-c+1$, 至少差1)
(比例上差不大, $a \neq c$, 至少差1, 误差比例上很大, 渐6)

②有 $b = q(c^2+1) - a$

知 $c(c^2-c+1) = p(aq(c^2+1) - a^2)$ ③

$$\geq aq(c^2+1) - a^2$$

$$\geq a(c^2+1) - a^2$$

$$= a(c^2 - a + 1)$$

$$c(c^2 - c + 1) = pab \geq a^2, \text{ 知 } a^2 \leq c(c^2 + 1)^2$$

故 $a \leq \frac{1}{2}(c^2+1)$, 显然, $c \leq \frac{1}{2}(c^2+1)$

及 $c(c^2-c+1) \geq a(c^2-a+1)$ 知 $a \leq c$

由 ③ $pa^2 = apq(c^2+1) - c(c^2-c+1)$

注意到 $apq(c^2+1) > c(c^2-c+1) = (c-1)(c^2+1)+1$

知 $apq > c-1$ 即 $apq \geq c$

进而 $pa^2 = apq(c^2+1) - c(c^2-c+1)$

$$= acpq + apq(c^2-c+1) - c(c^2-c+1)$$

$$\geq acpq$$

故 $a \geq qc \geq c$, 又 $a \leq c$, $q=1$

代入 ③, $p=1$, $c=c^2-c+1$, #

8. a, b, c 有两个相同显然, 下设两两不同

只需考虑 $a < b < c$ 情形

此时 b^a 相当大, a, b 较大时, a^b 远小于 b^a

n^{n+1} 是 n^n 的 n 倍, $(n+1)^n$ 仅是 n^n 的约 e 倍

引理: $n \in \mathbb{N}^*$, 有 $2 \leq (1+\frac{1}{n})^n < 3$

证明: 二项式定理

$$2(1+\frac{1}{n})^n = 1 + \frac{1}{n} \cdot n + \sum_{k=2}^n \frac{C_n^k}{n^k}$$

$$< 1 + 1 + \sum_{k=2}^n \frac{1}{k!}$$

$$\leq 2 + \sum_{k=2}^n \frac{1}{2^{k-1}} < 3. \#$$

$$\text{由 } a^c + b^a + c^b = a^b + b^c + c^a < a^b + c^b + a^c$$

$$\text{知 } b^a < a^b + c^a$$

$$\frac{b^a}{c^a} > \left(\frac{c+1}{c}\right)^{c+2} > 2 \text{ 由引理}$$

$$\frac{b^a}{a^b} = \frac{b^{a-b}}{\left(1+\frac{1}{a-1}\right)^b \left(1+\frac{1}{a-2}\right)^b \left(1+\frac{1}{b}\right)^b} > \left(\frac{b}{3}\right)^{a-b} \text{ 由引理}$$

$$b \geq 5 \text{ 时 } \frac{b^a}{a^b} < \frac{1}{2}, \frac{a^b}{b^a} > \frac{1}{2}, a-b=1, \text{ 故 } b=5$$

$$a^b < \frac{3}{5}b^a, c^a > \frac{3}{5}b^a$$

$$4^b > \frac{2}{5} \cdot 5^b, \left(1+\frac{1}{4}\right)^b < \frac{5}{2}, \text{ 矛盾}$$

$b=4, 3$, 不难类似知无解.

$$b=2 \text{ 时}, c=1, 2^a = a^2 - a + 2, a=3 \quad \#$$