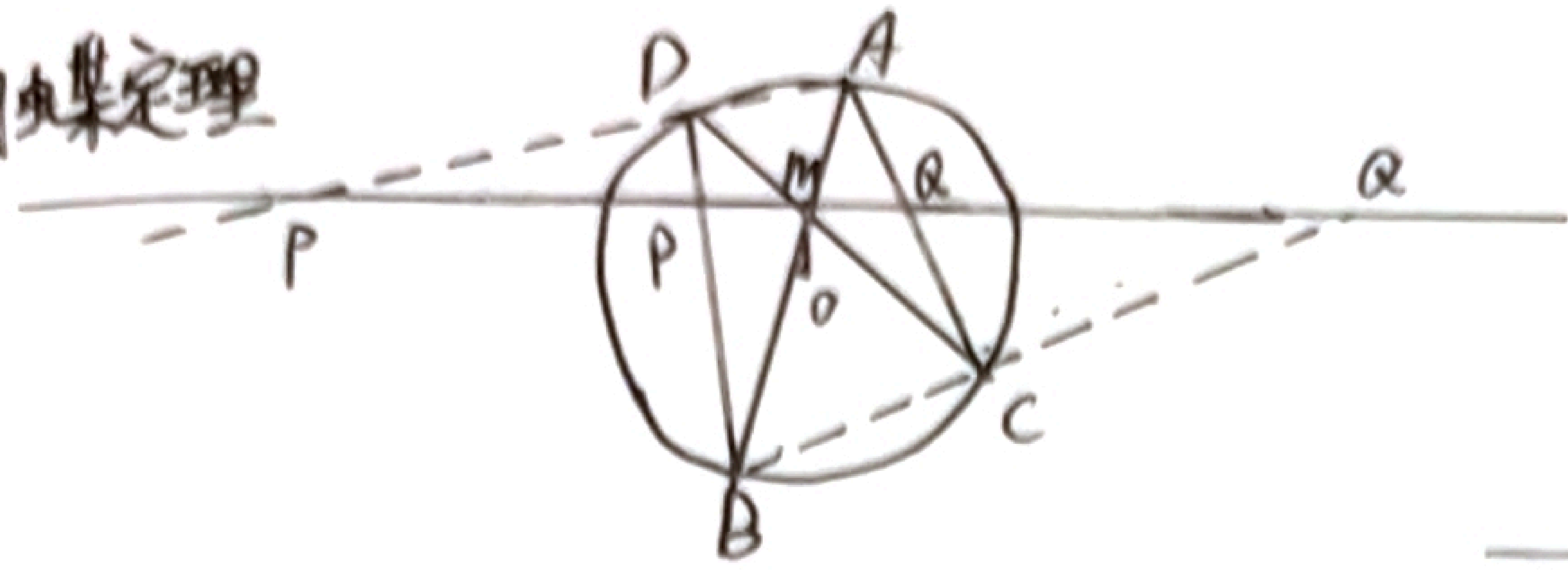
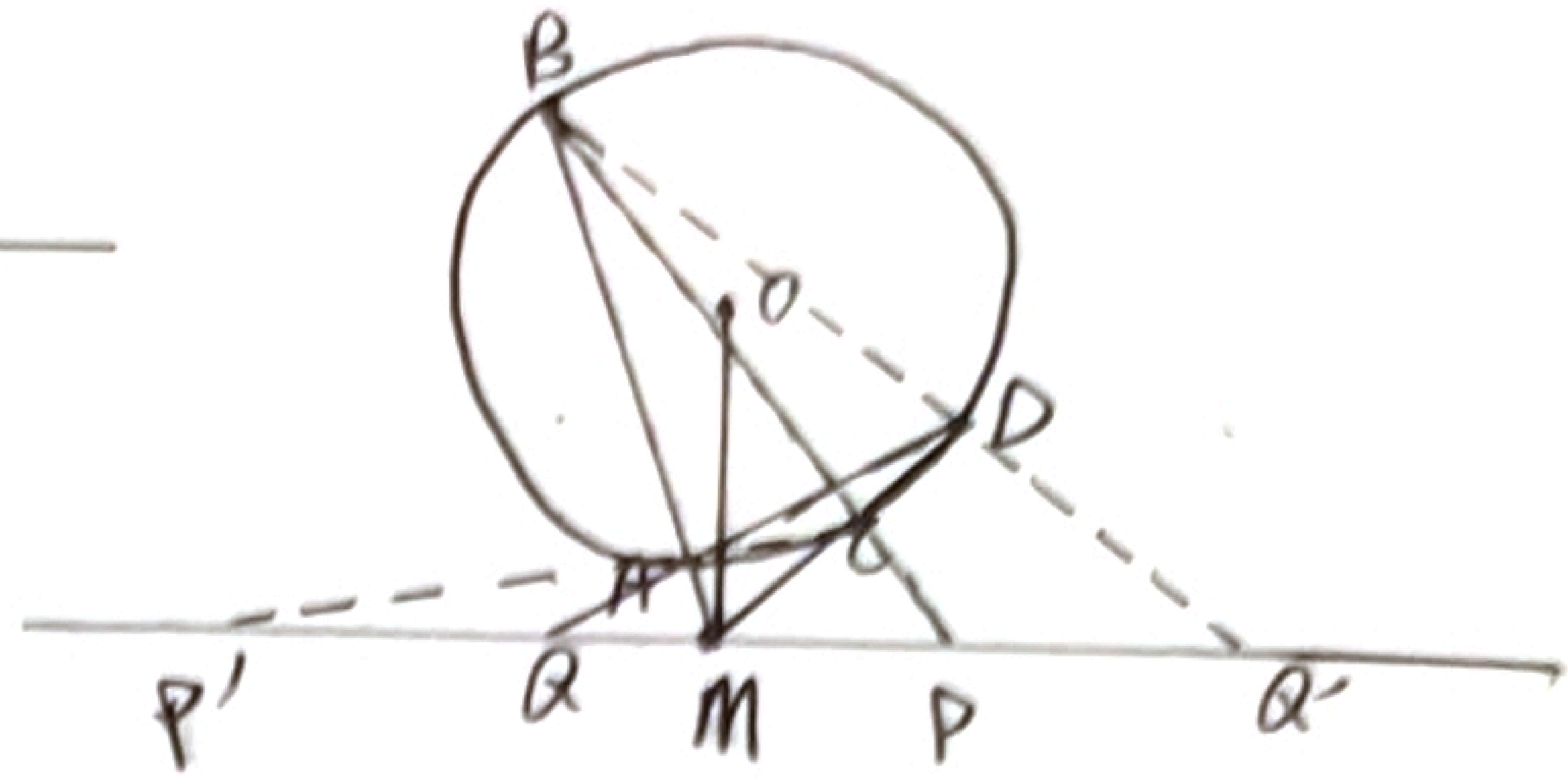
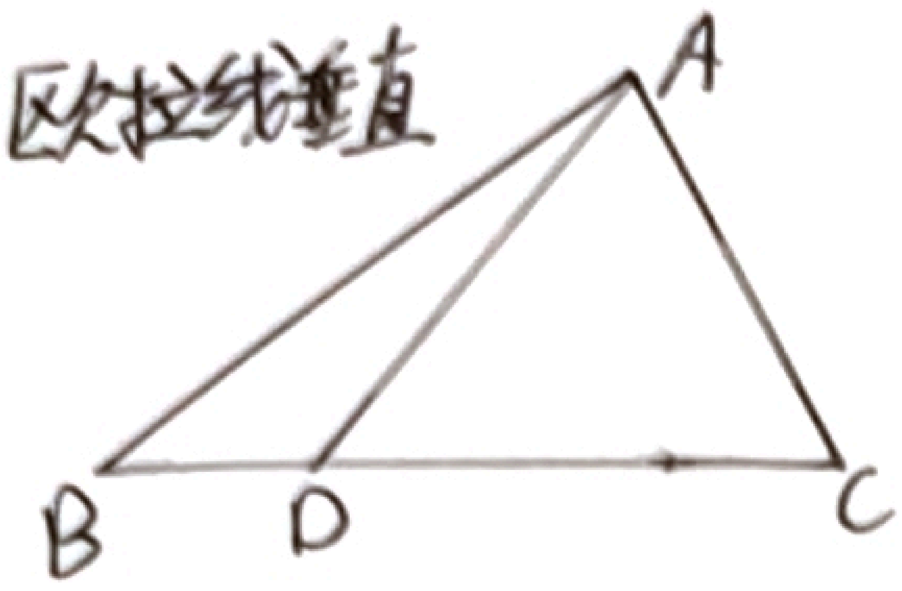
1. 蝴蝶定理





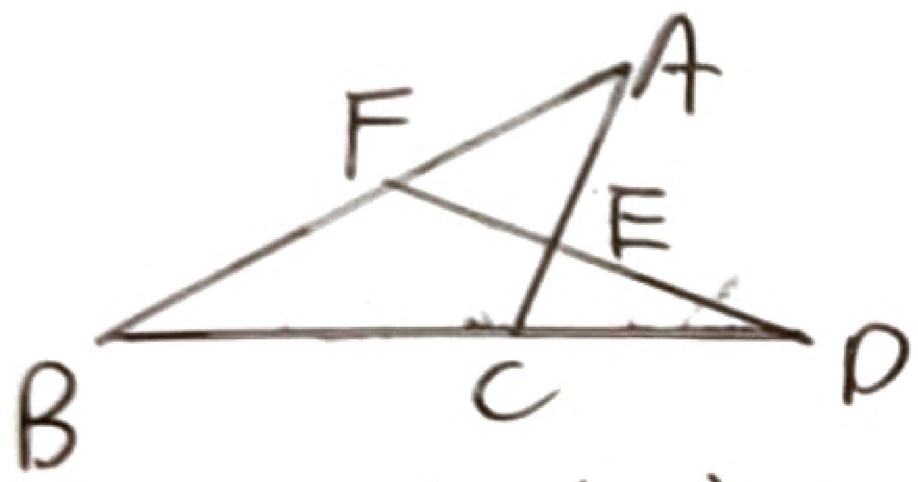
$PM=QM$

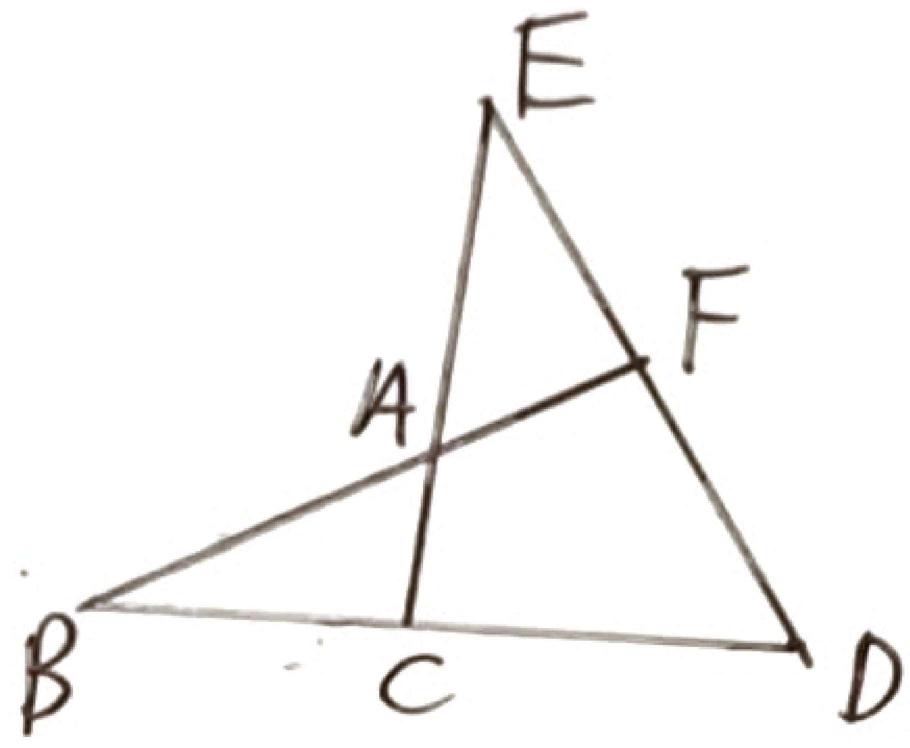
2. 欧拉线垂直



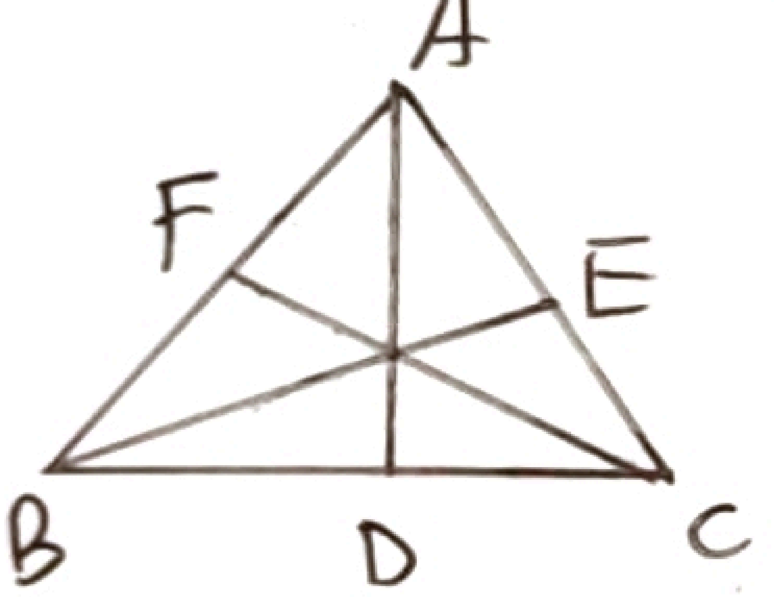
$\triangle ACD$与$\triangle ABD$的欧拉线垂直

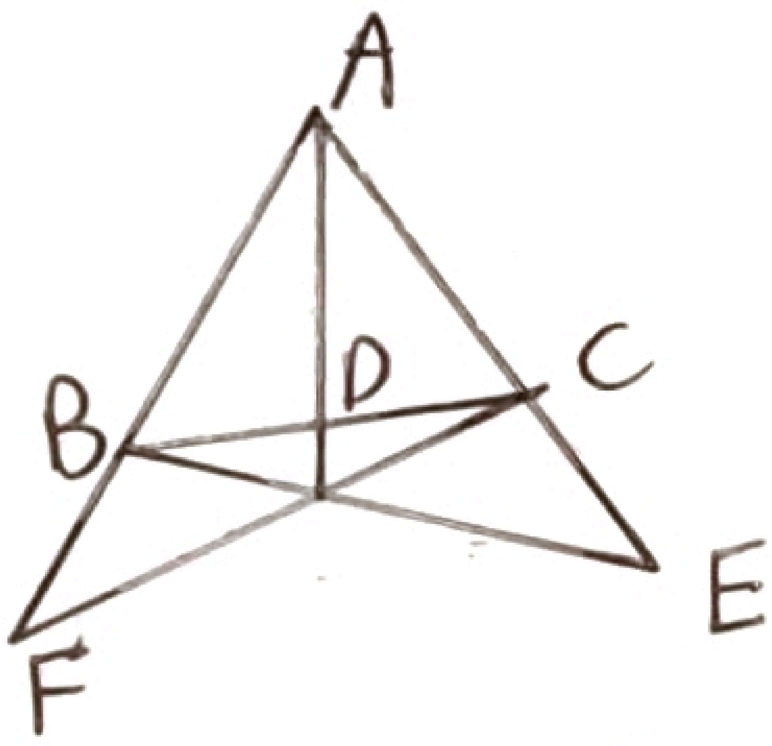
3. Menelaus定理和Ceva定理

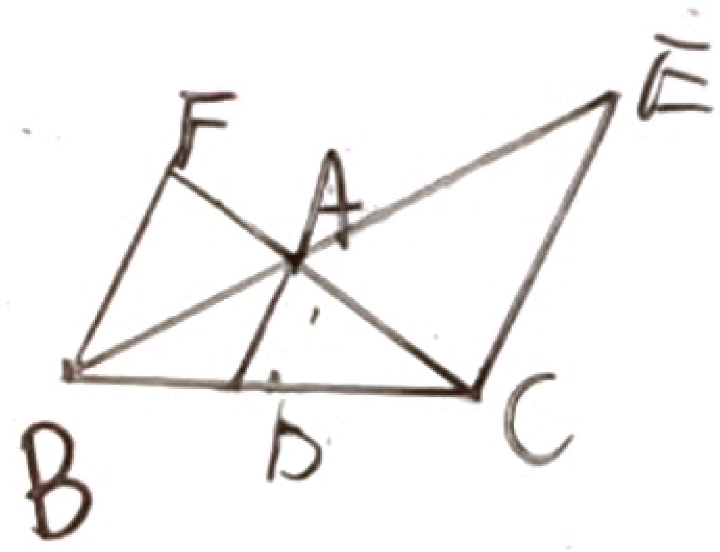




Menelaus定理: D、E、F共线 $\Leftrightarrow \frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1$　注: $\frac{\overline{BD}}{\overline{DC}}\cdot\frac{\overline{CE}}{\overline{EA}}\cdot\frac{\overline{AF}}{\overline{FB}}=-1$

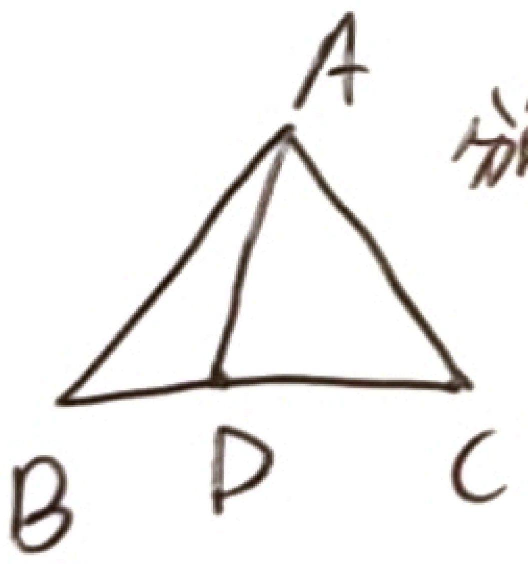






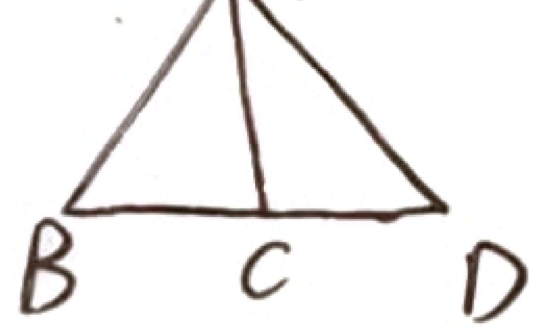
Ceva定理: AD、BE、CF共线 $\Leftrightarrow \frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=1$　注: $\frac{\overline{BD}}{\overline{DC}}\cdot\frac{\overline{CE}}{\overline{EA}}\cdot\frac{\overline{AF}}{\overline{FB}}=+1$

4.



偏线定理:

$$\frac{BD}{DC}=\frac{AB}{AC}\cdot\frac{\sin\angle BAD}{\sin\angle CAD}$$



5.

恒等式: $\frac{BD}{DC}\cdot\frac{CE}{EA}\cdot\frac{AF}{FB}=\frac{\sin\angle BAD}{\sin\angle DAC}\cdot\frac{\sin\angle CBE}{\sin\angle EBA}\cdot\frac{\sin\angle ACF}{\sin\angle FCB}$

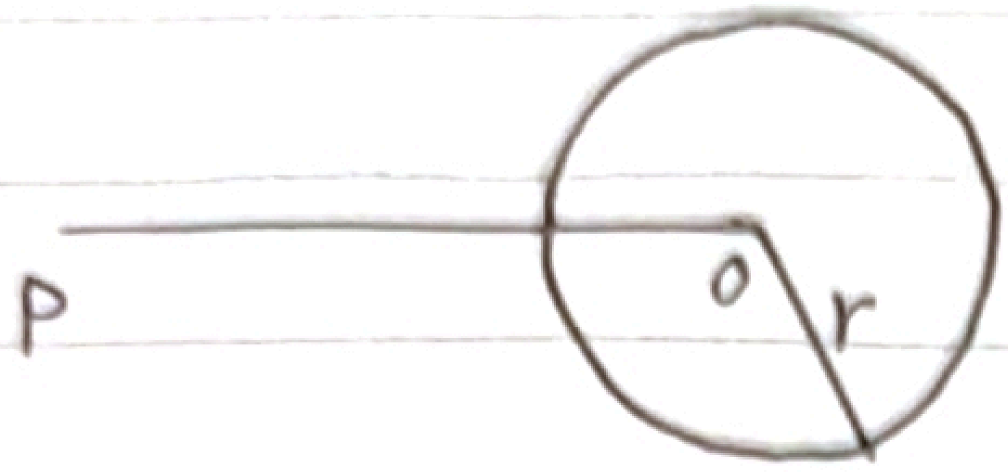
注: 可改为有向

故三角形式的Menelaus定理和Ceva定理:

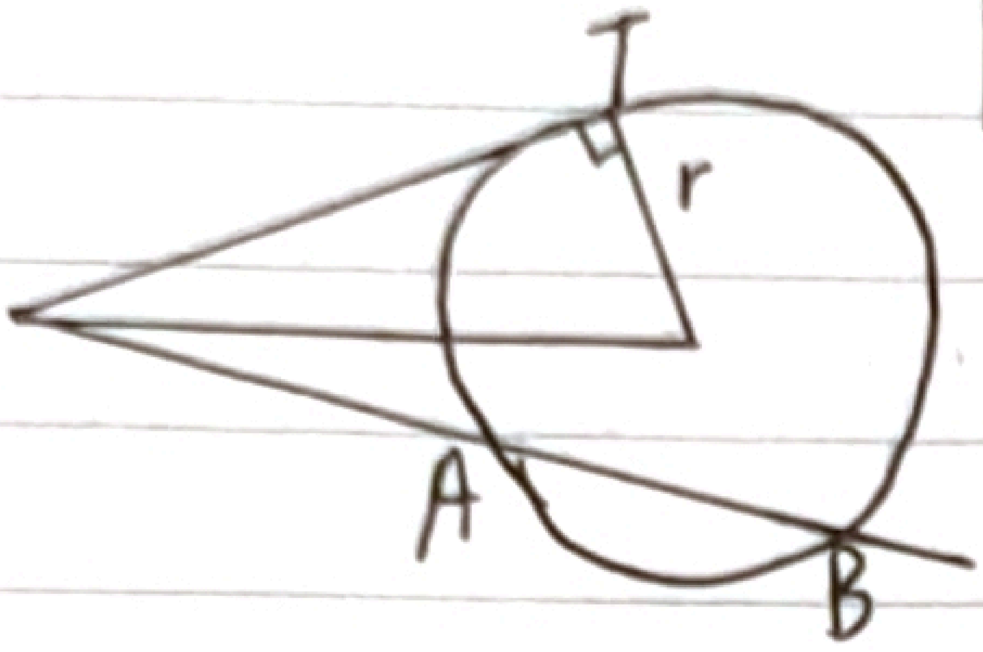
$$\frac{\sin\angle BAD}{\sin\angle DAC}\cdot\frac{\sin\angle CBE}{\sin\angle EBA}\cdot\frac{\sin\angle ACF}{\sin\angle FCB}=\mp1$$

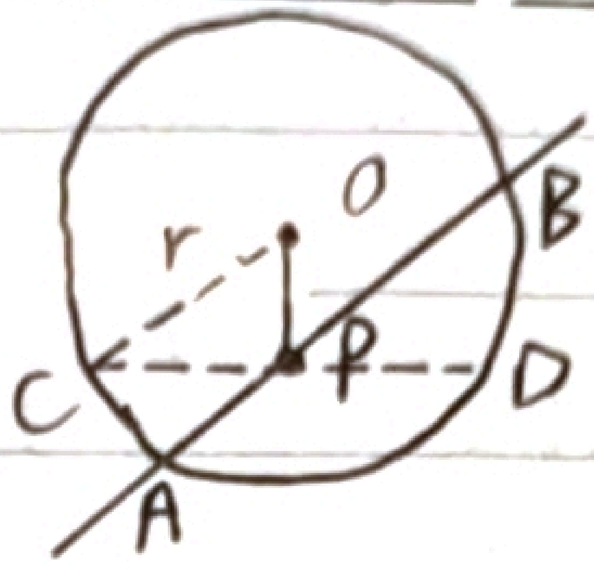
$$\frac{\sin\angle BOD}{\sin\angle DAC}\cdot\frac{\sin\angle COE}{\sin\angle EOA}\cdot\frac{\sin\angle AOF}{\sin\angle FOB}=\mp1$$

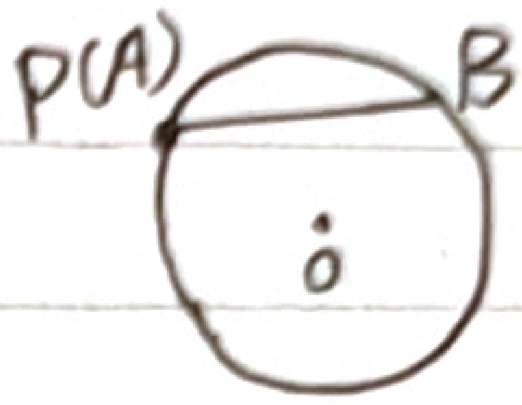
点对圆的幂



点P对圆O的幂为 $\rho(P)=PO^2-r^2$	
点P在圆外，$\rho(P)>0$	1. $\rho(P)=\rho(Q) \Leftrightarrow OP=OQ$
在圆上，$\rho(P)=0$	2. 给定圆γ与点P，过点P任作圆γ的一条
在圆内，$\rho(P)<0$	割线AB(A、B在圆上)，则 $\overline{PA}\cdot\overline{PB}=\rho(P)$ 为常量







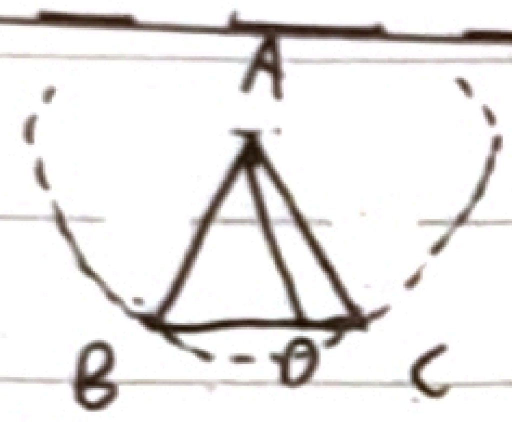
P在圆外时，$PA\cdot PB=\rho(P)$

P在圆内时，$-PC^2=-PC\cdot PD=\rho(P)=-PA\cdot PB$

$\overline{PA}\cdot\overline{PB}=\rho(P)$

P在圆上时，$PA=PA\cdot PB=0=\rho(P)$

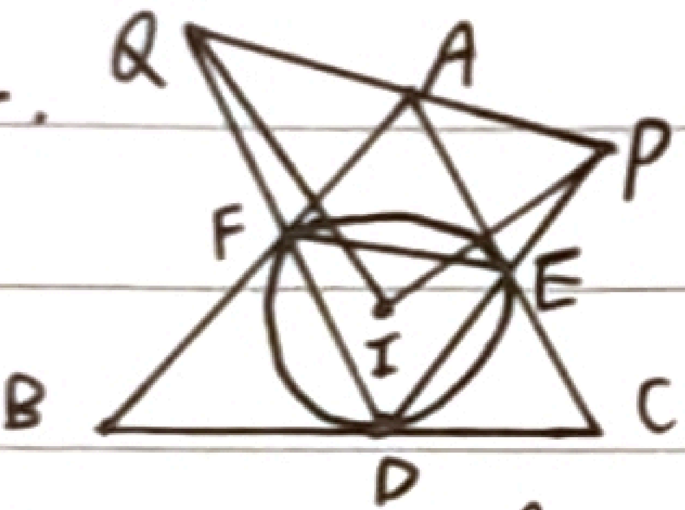
例一.



$AB=AD$

$\rho(D)=AD^2-AB^2=\overline{DB}\cdot\overline{DC} \Rightarrow AB^2-AD^2=DB\cdot DC$

例二.

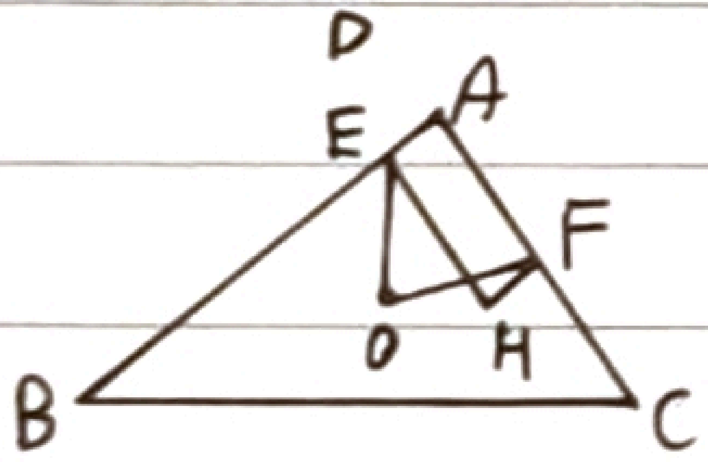


其中 $PQ\parallel EF$，求证 $\angle PIQ<90^\circ$

$PA\cdot PQ=PD\cdot PE=IP^2-r^2<IP^2$，$QA\cdot QP=QF\cdot QD=IQ^2-r^2<IQ^2$

$PA\cdot PQ+QA\cdot QP=PQ\cdot(AQ+AP)=PQ^2<IP^2+IQ^2$

例三

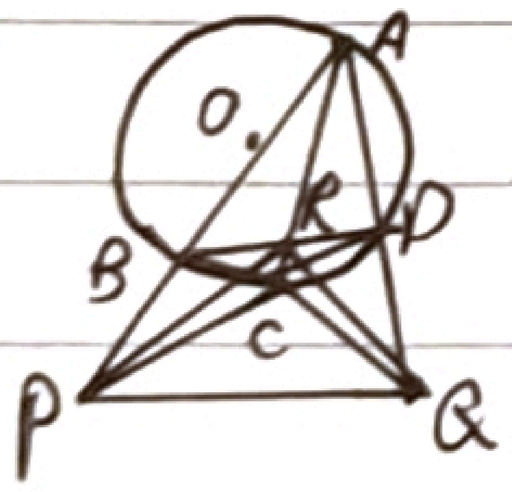


其中H为垂心，O为外心，A、E、H、F构成平行四边形，求证 $OE=OF$

$\Leftrightarrow \rho(E)=\rho(F) \Leftrightarrow EA\cdot EB=FA\cdot FC \Leftrightarrow HF\cdot EB=HE\cdot FC$

$\Leftrightarrow \frac{EB}{HE}=\frac{FC}{HF} \Leftrightarrow \triangle HEB\sim\triangle HFC \Leftrightarrow \angle HBA=\angle ACH$ ✓

例四



布洛卡定理

$\rho(P)+\rho(Q)=PQ^2$，$\rho(P)+\rho(R)=PR^2$，$\rho(Q)-\rho(R)=PQ^2-PR^2$

$QO^2-OR^2=PQ^2-PR^2$，$OP\perp QR$，∴O是垂心

两圆的根轴直线上任意点对⊙O_1、⊙O_2的幂相等，l即为根轴

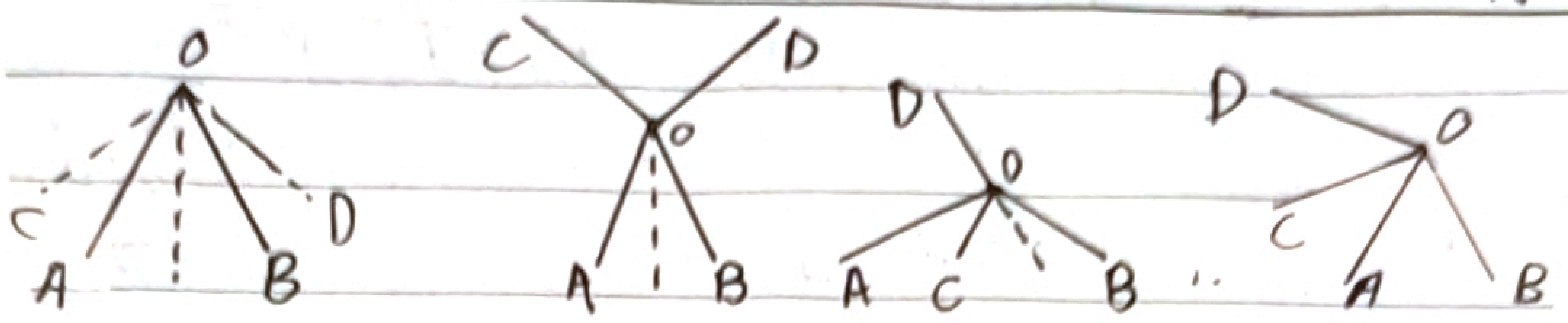
$PO_1^2-r_1^2=PO_2^2-r_2^2$，$PO_1^2-PO_2^2=r_1^2-r_2^2$

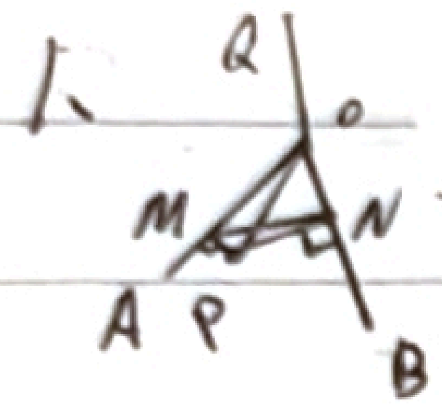
可用于证三点共线

根心：三圆，两两形成三根轴，交于同一点。可用于证三线共点.

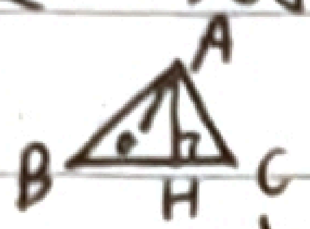
等角线

OP平分$\angle AOB$，OC与OD关于OP对称，OD为OC关于$\angle AOB$的等角线

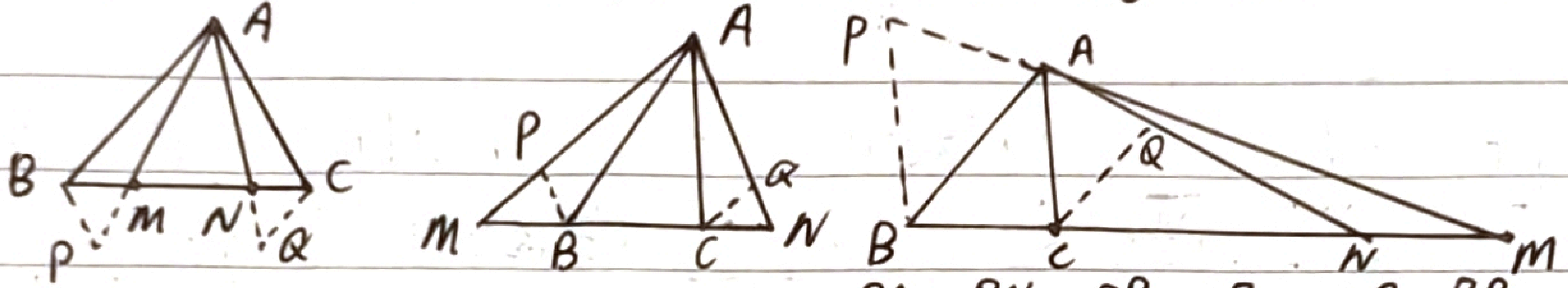


1. 

OQ是OP关于$\angle AOB$的等角线 $\Leftrightarrow$ $OQ \perp PN$

例  $\angle BAO$是$\angle CAH$关于$\angle BAC$的等角线　即顶点与外心连线的等角线是[illegible]

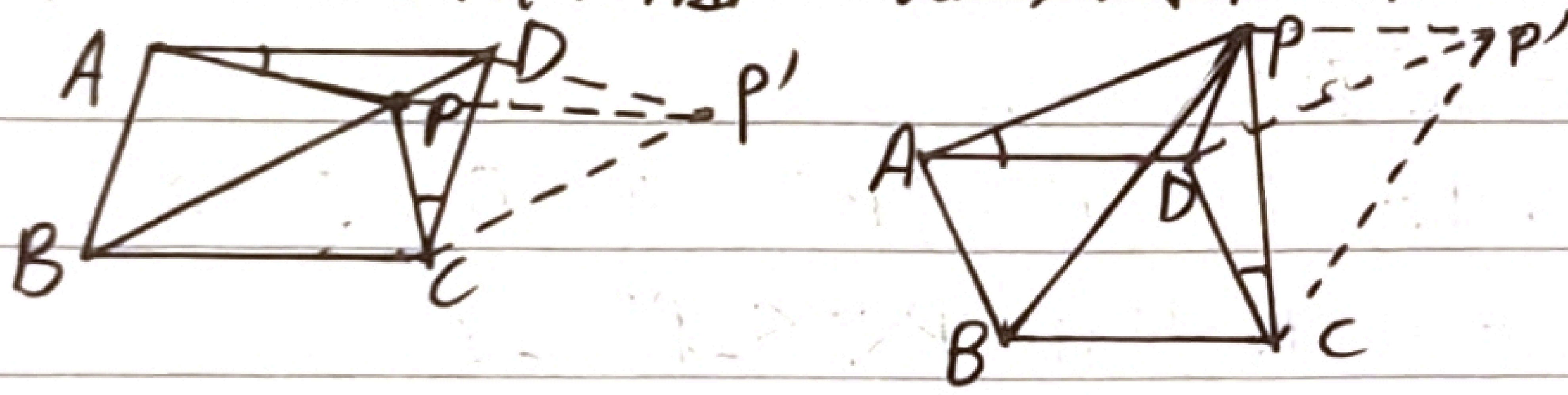
2. AM、AN关于$\angle BAC$互为等角线 $\Longleftrightarrow$ $\frac{BM}{MC}\cdot\frac{BN}{NC}=\frac{AB^2}{AC^2}$



其中$BP /\!/ AC$，$CQ /\!/ AB$

证明：$\frac{BM}{MC}\cdot\frac{BN}{NC}=\frac{BP}{AC}\cdot\frac{AB}{CQ}=\frac{AB}{AC}\cdot\frac{BP}{CQ}$

3. P是▱$ABCD$所在平面上一点，则PB、PD关于$\angle APC$互为等角线 $\Longleftrightarrow$ $\measuredangle PAD=\measuredangle DCP$



注：方向角模180°后等价

4. 完全四边形等角线性质