2. $\odot(ABCDE)$

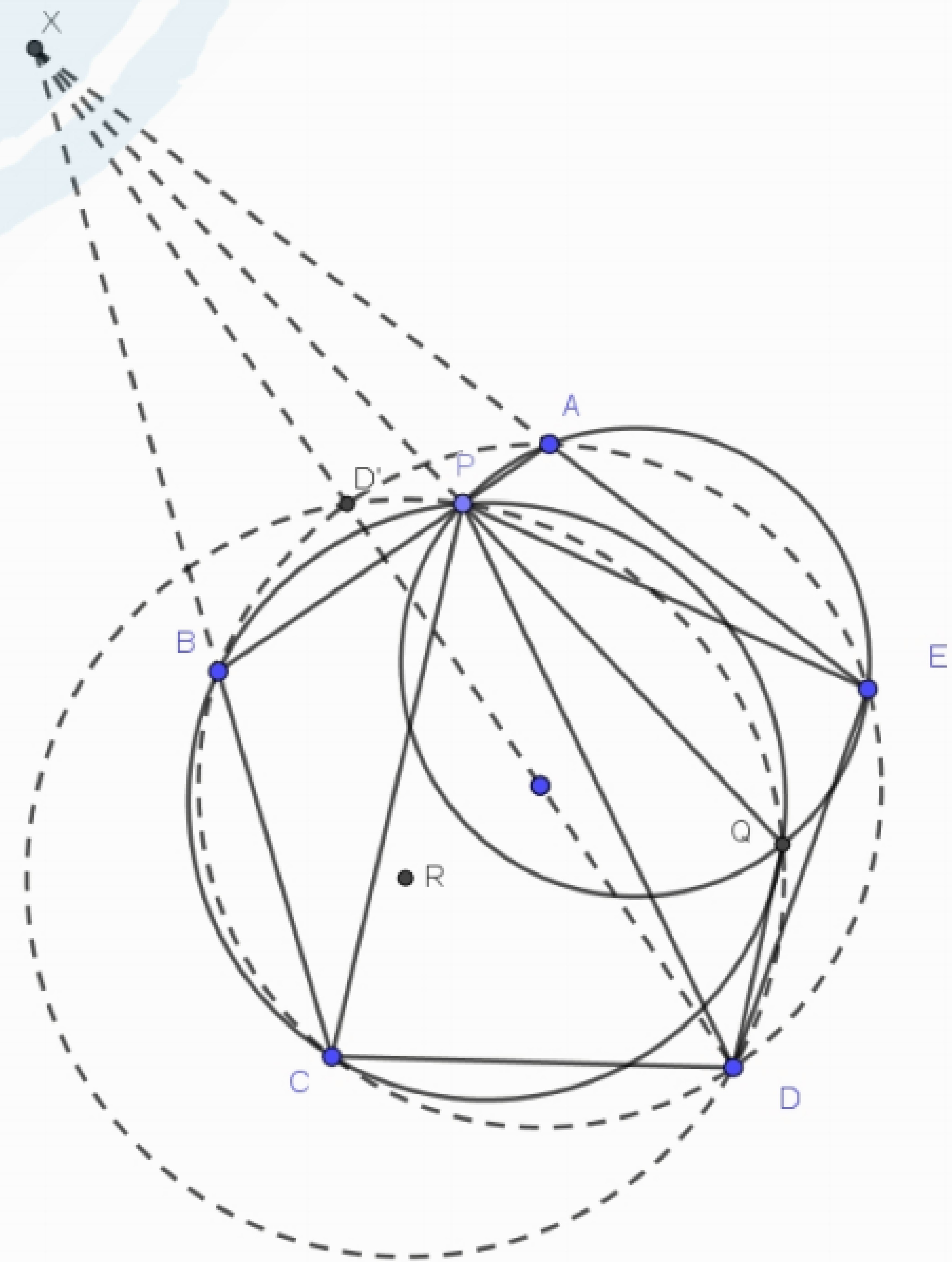
$\odot(BCP)$

$\odot(AEP)$

PQ AE, BC 交于一点 X

$XP \cdot XQ = XD' \cdot XD$.

R 在 DD' 中垂线上.



1.

$$\frac{x_1+x_2}{x_2+x_3}+\cdots+\frac{x_n+x_1}{x_1+x_2}\geq n \quad (A-G)$$

$$\frac{x_2}{x_2+x_3}+\cdots+\frac{x_1}{x_1+x_2}\geq\frac{n}{2}$$

$$\Leftrightarrow\quad \frac{x_3}{x_2+x_3}+\cdots+\frac{x_2}{x_1+x_2}\leq\frac{n}{2}$$

$$\Leftrightarrow \frac{x_1-x_2}{x_1+x_2}+\cdots+\boxed{\frac{x_n-x_1}{x_n+x_1}}\geq 0. \quad \checkmark$$

$f(x_i) \quad \to x_{i+1} / x_{i-1}$

$$\frac{1}{1+\frac{x_2}{x_1}}+\frac{1}{1+\frac{x_3}{x_2}}+\cdots+\frac{1}{1+\frac{x_n}{x_{n-1}}}+\frac{1}{1+\frac{x_1}{x_n}}\geq\frac{n}{2}$$

Tensen

$n=3$. $(x_1-x_2)(x_1-x_3)(x_2-x_3) \geq 0$.

$n-1$ ✓. $\underline{n}$

$$\frac{x_1}{x_1+x_2}+\frac{x_2}{x_2+x_3} \geq \frac{1}{2}+\frac{x_2}{x_1+x_3} \rightarrow x$$

$\underline{\cancel{n-1}}$

$$\frac{\quad \cdots \quad x_n}{x_1 \cdot x_3 x_4 \quad \cdots \quad x_n}$$

$x_1 x_2 x_3$

$$\sum \frac{x_1 - x_2}{x_2 + x_3} \geq 0$$

$$\frac{x_1 - x_2}{x_2 + x_3} + \cdots + \frac{x_{n-1} - x_n}{x_n + x_1} \geq \frac{x_1 - x_n}{x_1 + x_2}$$

$$k \geq -1$$

4. $\frac{2^n+1}{n+2} \geq \sum_{p} (p-1) v_p(d(2^n+1))$

$d(x)$ 表示 x 的因子个数.

$$f(x) = \sum_{p} (p-1) v_p(x)$$

$$f(x) + f(y) = f(xy).$$

$$f(p_1^{\alpha_1} \cdots p_s^{\alpha_s}) = \alpha_1 f(p_1) + \alpha_2 f(p_2) + \cdots + \alpha_s f(p_s)$$

$f(p)=p-1.$

$d=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_s^{\alpha_s}.$

$f(d)=\sum_{i=1}^{s}\alpha_i(p_i-1)\leq\frac{2^n+1}{n+2}$

$d=d(2^{2^n}+1).$ $\boxed{d\leq N}$

引: $2^{2^n}+1$素因子 $p\equiv 1(2^{n+2}).$

$p\equiv 1(2^{n+1})$

$a = 2^{2^{n-2}}(2^{2^{n-1}}-1)$.

$a^2 \equiv 2\ (p)$

由于 $p \mid 2^{2^{n+1}}-1$. $\delta_p(2) = 2^{n+1}$.

表明 $\delta_p(a) = 2^{n+2}$, $\to$ $p \equiv 1\ (2^{n+2})$

$\underline{p \mid 2^{2^n}+1} \Rightarrow \underline{p \mid 2^{2^{n+1}}-1}$. $\mid \underline{p \mid 2^{\delta_p(2)}-1}$

$\delta_p(2)$. $2^{\delta_p(2)} \equiv 1\ (p)$ min $\delta_p(2)$.

$$p \mid 2^{(\delta_p(2),\, 2^{n+1})} - 1 \qquad p \nmid 2^{2^n} - 1$$

$$(\delta_p(2),\, 2^{n+1}) = 2^{n+1} \Rightarrow \delta_p(2) = 2^{n+1}$$

$$p \mid 2^{p-1} - 1 \qquad p \mid 2^{\gcd(2^{n+1},\, p-1)} - 1$$

$$2^{n+1} \mid p-1$$

$p = 2^{n+1}+1.$ $\qquad$ $p \geq 2^{n+2}+1$

$p \nmid 2^{2^n}+1$ $\qquad$ $p \mid 2^{2^{n+1}}-1.$

$2^{n+1}+1 \mid 2^{2^{n+1}}-1.$ $\qquad$ $2^{n+1}+1 \mid 2^{2(n+1)}-1$

$2^{n+1}+1 \mid 2^{\gcd(2^{n+1},\, 2(n+1))}-1$

$2(n+1) = 2^{\alpha}.$ $\qquad$ $n = 2^{\alpha-1}-1$

$$\underline{2^{n+1}+1} = (\underbrace{2^{2^{\alpha-1}}+1}, \overline{2^{2^n}+1}) = 1$$

$$2^{2^{\alpha-1}}+1 \mid 2^{2^n}-1.$$

$$\underline{p \geq 2^{n+2}+1}$$

$$2^{2^n}+1 = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_s^{\alpha_s} \qquad p_i \geqslant 2^{n+2}+1$$

$$d = (\alpha_1+1) \cdots (\alpha_s+1) = q_1^{\beta_1} \cdots q_t^{\beta_t}.$$

$$f(d) = ? \qquad f(d) \le \frac{2^n+1}{n+2}$$

$$f(d) = f(\alpha_1+1) + \cdots + f(\alpha_s+1) \lesssim \alpha_1 + \cdots + \alpha_s.$$

$$\alpha_1 + \cdots + \alpha_s \le \log_{2^{n+2}+1}(2^{2^n}+1).$$

$$\le \log_{2^{n+2}}(2^{2^n}+1) = \frac{2^n+1}{n+2}$$

$$\frac{f(x+1)\le x.}{f(p)=p-1.}$$

$$f(xy)=f(x)+f(y)$$

归纳: 1 ✓.

$\le k$. ✓.

$k+1$.

① $k+1$素数 $f(k+1)=k$ ✓

② $k+1$合数 $k+1=pq$. $p\geqslant 2, q\geqslant 2$

$$f(k+1) = f(p)+f(q) \leq p-1+q-1$$

$$p-1+q-1 \leq k = pq-1.$$

$$\Leftarrow (p-1)(q-1) \geq 0.$$

3.

$$A = \{a_1, \cdots a_k\}.$$

$$A + A \geq \{\underline{1^2, 2^2 \cdots n^2}\}.$$

$$|A| > n^{0.666}. \qquad (n \to +\infty) \qquad |A| > n^{\frac{2}{3} - \varepsilon}.$$

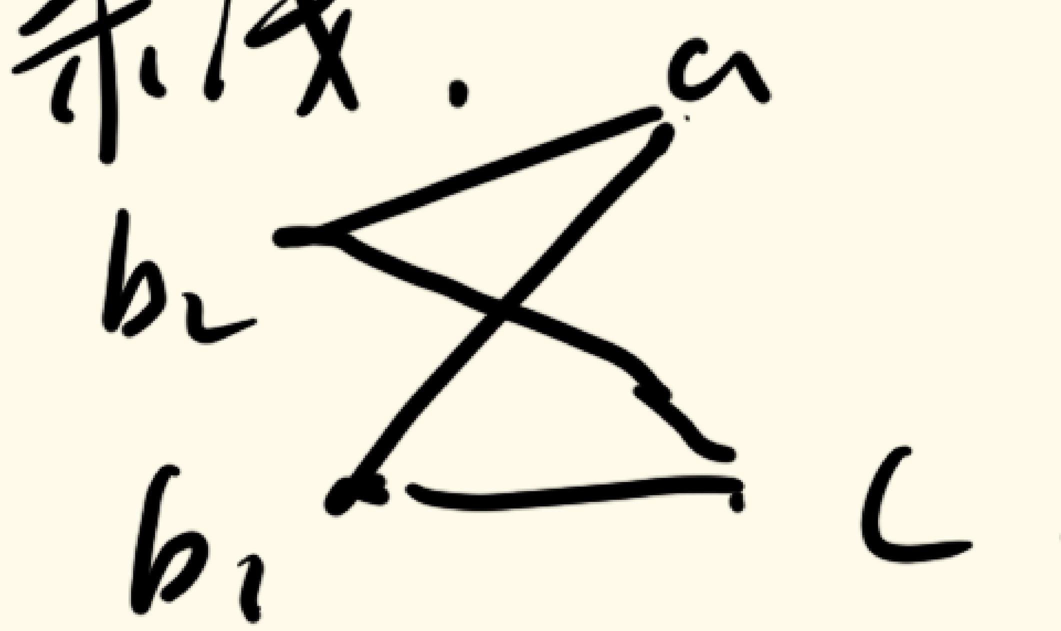
$$|A|^2 \geq n \qquad |A| \geq n^{0.5}$$

图 $G(V;E)$.

$V=A$. E 如下决定:

若 $a+b\in\{1^2,2^2\cdots n^2\}$. $(a\neq b)$.

则 $a-b$ 连一条线. 至少有 $n-|A|$ 条线.



$|E|\geq n-|A|$

考虑顶点对 (a,c). 满足 (a,b_1,c) (a,b_2,c) $\cdots$ (a,b_k,c).

$a-b_1-c$, $a-b_2-c$ $\cdots$ $a-b_k-c$

形成角.

顶点对有 $C_{|A|}^2$ 对.

表示至少有1对顶点满足：它们属至少有

$\frac{S}{C_{|A|}^2}$ S表示总角个数.

设 $V=\{a_1 \cdots a_{|A|}\}$

a_i度为 d_i $\quad \sum_{i=1}^{|A|} d_i = 2|E|$

角个数 $\sum_{i=1}^{|A|} C_{d_i}^2$

$$\sum_{i=1}^{|A|} C_{a_i}^2 \geq |A| \, C^2_{\frac{2|E|}{|A|}} = S_0$$

$w(a,c)$ 顶点对 → 有多少

$\frac{S_0}{C^2_{|A|}}$ 个角 若 $|A| < n^{0.666}$

$$S_0 \approx \frac{1}{2}\left(\frac{2|E|}{|A|}\right)^2 |A| \approx 2 \frac{|E|^2}{|A|}$$

$$\approx 2 \frac{n^2}{n^{0.666}} \approx 2 \cdot n^{1.334}$$

$$C_{|A|}^2 \approx \frac{1}{2}|A|^2 \approx \frac{1}{2}\cdot n^{1.332}$$

导出

$(c, a) \longrightarrow (c, b, a)$ 角

至少有 $c \cdot n^{0.002}$ 个.

$$c+b_i=s_i^2$$

$$a+b_i=t_i^2$$

$c-a=s_i^2-t_i^2$ 至少 $n^{0.002}$ 个解

$$c-a=(s_i-t_i)(s_i+t_i)$$

$c-a$ 的一组因数决定一组 (s_i,t_i)

这里 (s_i,t_i) 解数 $\le d(c-a)$.

显然一点，我们可以去掉所有 $>n^2$ 元素，

不妨设 $c, a \leq n^2$

$|c-a| \leq n^2$

这表明 $d(|c-a|) \geq \lambda \cdot n^{0.002}$ 个. 对充分大 n 不成立.

矛盾！证毕！

对充分大的 n. 我们证明: $\forall \lambda>0$.

$1\le S\le n$.

$d(S) < \lambda \cdot n^{0.001}$. $\quad d(S)<\lambda; n^{0.0009}$.

$S = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$

$d(S) = (\alpha_1+1)(\alpha_2+1)(\cdots)(\alpha_k+1)$. $(\alpha_i \geq 0)$

$\ln S = \sum_{i=1}^{k} \alpha_i \ln p_i$.

$\ln(d(S)) = \sum_{i=1}^{k} \ln(\alpha_i+1)$

证明：n 充分大时.

$$\frac{\ln d(s)}{\ln s} \le 0.0009.$$

找到. $\frac{\ln(x+1)}{x} \le 0.0008$ 对 $x > 10^{100}$ 成立.

$$\frac{1}{\ln x} \le 0.0008 \text{ 对 } x > 10^{10^{100}} \text{ 成立.}$$

$$\frac{\ln d(s)}{\ln s} = \frac{\sum_{p_i < 10^{10^{100}}} \ln(\alpha_i+1) + \sum_{p_i > 10^{10^{100}}} \ln(\alpha_i+1)}{\sum_{p_i < 10^{10^{100}}} \alpha_i \ln p_i + \sum_{p_i > 10^{10^{100}}} \alpha_i \ln p_i}$$

$$\leq \frac{\sum_{p_i<} \ln(\alpha_i+1) + 0.0008\,T.}{\sum_{p_i<} \alpha_i + T}$$

$$\leq \frac{\sum_{p_i<\alpha_i<} \ln(\alpha_i+1) + 0.008\,T_1}{\sum_{p_i<\alpha_i<} \alpha_i + T_1}$$

$$\le \frac{0.0008T_1 + T_0.}{T_1}$$

$$T_1 \to +\infty.$$

$$\frac{\ln(d(s))}{\ln s} \to 0.088$$

存在常数 C，使对任意 $n\in\mathbb{Z}_+$，

$$\tau(n)\le Cn^{\frac{1}{1000}}$$

$$n=p_1^{\alpha_1}\cdots p_k^{\alpha_k}$$

$$\frac{\tau(n)^{1000}}{n}=\prod_{i=1}^{k}\frac{(\alpha_i+1)^{1000}}{p_i^{\alpha_i}}$$

对 $p>2^{1000}$，$(\alpha+1)^{1000}\le p^{\alpha}$.

对 $p<2^{1000}$. $p\geqslant 2$

$$\frac{(\alpha+1)^{1000}}{2^{\alpha}} \to 0 \quad (\alpha \to +\infty)$$

所以存在一个常数M.

$$\frac{(\alpha+1)^{1000}}{p} \leq \frac{(\alpha+1)^{1000}}{2^{\alpha}} \leq M.$$

从而 $\frac{\tau(n)^{1000}}{n} \le \prod_{p \le 2^{1000}} \frac{(\alpha_i+1)^{1000}}{p_i^{\alpha_i}} \le M^{\pi(2^{1000})}$